THE CAMBRIDGE

MATHEMATICAL JOURNAL.

VOL. II.] MAY, 1840. [No. IX.

I.—ON CERTAIN THEOREMS IN THE CALCULUS OF VARIATIONS.

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IT would perhaps have been more just to entitle this communication, "Notes on Lagrange." The papers from which it is selected were written towards the close of the year 1838, during the perusal of the *Mécanique Analytique*. Every mathematician is aware of the important uses which the illustrious author has there made of the Calculus of Variations. The mode in which it is employed as an instrument of demonstration, consists in almost every instance in a comparison of the terms of developed expressions, a mode of investigation by which the abstract existence of truths is shown, much more clearly than the nature of their mutual dependence. Now as the results obtained presuppose, in a majority of instances, the fulfilment of a certain primitive condition, (the integrability of $Xdx + Ydy + Zdz$,) it appeared to the writer of this paper, that a mode of demonstration, which should establish in a more direct way the connection between the above condition, and the great secondary principles of dynamics thence deducible, would in many respects possess the advantage. From this consideration the following attempts originated.

As the basis of these investigations, I suppose the symbols d and δ to imply two independent differentiations, performed on one variable quantity, supposed to be a function of two others. This being laid down, it is easy to establish the following principles:—

1st. The symbols, $\int$, d, δ, are mutually transposable, together with the symbols $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{\delta}{\delta x}$, $\frac{\delta}{\delta y}$, &c., when the denominators of these latter are relatively constant.

2nd. If U and V represent functions of u, then

$$dU\,\delta V = dV\,\delta U.$$

The last-named includes implicitly the following important cases:

1st. U being a function of u, $\frac{dU}{du} = \frac{\delta U}{\delta u}$.

2nd. P being a function of x, y, z,

$$\delta P = \frac{dP}{dx}\delta x + \frac{dP}{dy}\delta y + \frac{dP}{dz}\delta z.$$

The application of the Calculus of Variations to the determination of the maximum or minimum of an indeterminate integral formula $\int U dx$, in which x is the independent variable, and U a function of x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c., presents some difficulties which are not experienced, when U is a function of x, y, dx, dy, d^2x, d^2y, &c. The reason of this appears to be, that in the former case y and its differentials are susceptible of two different kinds of variation, one resulting from x, since y is a function of that variable, the other independent of x. The former of these variations I therefore suppose to arise from some quantity entering into the constitution of x, the latter from some other quantity entering with x into the constitution of y. This distinction leads to the following very simple investigation.

Let δ_1 represent all variation through the medium of x, δ' that which is independent of x. Then

$$\delta U = \frac{\delta_1 U}{\delta_1 x}\delta_1 x + \frac{dU}{dy}\delta' y + \frac{dU}{d\frac{dy}{dx}}\delta'\frac{dy}{dx} + \&c.$$

Now since x enters into every term,

$$\frac{\delta_1 U}{\delta_1 x} = \frac{1}{dx}dU.$$

Moreover, by the principle of transposition,

$$\delta'\frac{dy}{dx} = \frac{d}{dx}\delta' y, \quad \delta'\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}\delta' y, \ \&c.$$

Hence, on substitution,

$$\delta U = \frac{1}{dx}dU\,\delta x + \frac{dU}{dy}\delta' y + \frac{dU}{d\frac{dy}{dx}}\frac{d}{dx}\delta' y + \frac{dU}{d\frac{d^2y}{dx^2}}\frac{d^2}{dx^2}\delta' y + ..$$

If by pdx we represent that portion of δy which is derived from x, then $\delta' y = \delta y - p\delta x$, which substituted in the above will give Lagrange's expression.

Every circumstance of motion of a system of free bodies in space is included in the following equation, (*Mec. Anal.*)

$$\Sigma\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z + X\delta x + Y\delta y + Z\delta z\right) = 0 \ldots (A),$$

which may be resolved, in consequence of the independence of δx, δy, δz, into the triple system,

$$\frac{d^2x}{dt^2} = -X, \quad \frac{d^2y}{dt^2} = -Y, \quad \frac{d^2z}{dt^2} = -Z.$$

On changing in (A) δ into d, we have

$$\Sigma\left(\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz + Xdx + Ydy + Zdx\right) = 0 \ldots (B);$$

and it is supposed that $Xdx + Ydy + Zdz$ is an exact differential whose integral may be represented by V.

The integration of (B) gives

$$\Sigma\left(\frac{dx^2 + dy^2 + dz^2}{2dt^2} + V\right) = H \text{ (a constant)} \ldots\ldots (C).$$

This equation, usually presented under the form $T + V = H$, represents the conservation of the living forces of the system.

Taking the variation of the last expression, we have

$$\Sigma\left(\frac{dx\,d\delta x + dy\,d\delta y + dz\,d\delta z}{dt^2} + X\delta x + Y\delta y + Z\delta z\right) = 0.$$

whence, by comparison with (A),

$$\Sigma\,(dx\,d\delta x + dy\,d\delta y + dz\,d\delta z) = \Sigma\,(d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z).$$

To each side of this equation add the first side, and observing that each side becomes integrable, we shall have

$$\Sigma\delta\,(dx^2 + dy^2 + dz^2) = \Sigma d\,(dx\,\delta x + dy\,\delta y + dz\,\delta z);$$

whence

$$\delta\Sigma\int\left(\frac{dx^2 + dy^2 + dz^2}{dt}\right) = \Sigma\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right);$$

and if δx, δy, δz, be supposed to vanish for the extreme points, we have

$$\delta\int 2T\,dt = 0,$$

which expresses the principle of least action; an appellation, the propriety of which it does not fall within the scope of this paper to inquire into.

The function $\dfrac{dx^2 + dy^2 + dz^2}{2dt^2}$ being still represented by T, if for T – V we substitute Z, and suppose V converted into a function of ξ, ψ, ϕ, so that T and Z become functions of ξ, ψ, ϕ, ξ', ψ', ϕ', homogeneous with respect to the three last, (since the relation between the differential coefficients of the new and old variables is linear); then will the differential equations of motion become (*Mec. Analytique*, Sec. Partie v. §. 1.)

$$\left.\begin{aligned} d\frac{dZ}{d\xi'} - \frac{dZ}{d\xi}dt &= 0, \\ d\frac{dZ}{d\psi'} - \frac{dZ}{d\psi}dt &= 0, \\ d\frac{dZ}{d\phi'} - \frac{dZ}{d\phi}dt &= 0, \end{aligned}\right\} \ldots\ldots (D).$$

I shall now proceed to demonstrate a theorem, forming the basis of Lagrange's investigations on the great problem of the variation of the arbitrary constants, in questions of dynamics, using, as he has done, the symbol d to denote differentiation with respect to the time, and Δ and δ to imply variations relative to the arbitrary constants which may be supposed to enter into the constitution of ξ, ψ, ϕ.

Since Z is a function of ξ, ψ, ϕ, ξ', ψ', ϕ',

$$\delta Z = \frac{dZ}{d\xi}\delta\xi + \frac{dZ}{d\xi'}\delta\xi' + (\ ),$$

$$\Delta Z = \frac{dZ}{d\xi}\Delta\xi + \frac{dZ}{d\xi'}\Delta\xi' + (\ ),$$

( ) indicating the corresponding terms with respect to ψ, ψ', ϕ, ϕ'. Now $\Delta\delta Z = \delta\Delta Z$; whence, performing the requisite operations, and transposing to the first side,

$$\left.\begin{aligned} &\Delta\frac{dZ}{d\xi}\delta\xi + \frac{dZ}{d\xi}\Delta\delta\xi + \Delta\frac{dZ}{d\xi'}\delta\xi' + \frac{dZ}{d\xi'}\Delta\delta\xi' + (\ ) \\ &-\delta\frac{dZ}{d\xi}\Delta\xi - \frac{dZ}{d\xi}\delta\Delta\xi - \delta\frac{dZ}{d\xi'}\Delta\xi' - \frac{dZ}{d\xi'}\delta\Delta\xi' - (\ ) \end{aligned}\right\} = 0.$$

The even terms of this expression annul each other.

Moreover by (D), $\frac{dZ}{d\xi} = \frac{d}{dt}\frac{dZ}{d\xi'}$; and by the principle of transposition laid down at the commencement of this paper,

$$\delta\xi' = \delta\frac{d\xi}{dt} = \frac{d}{dt}\delta\xi, \quad \text{and} \quad \Delta\xi' = \frac{d}{dt}\Delta\xi.$$

Making these substitutions, and rejecting the common denominator dt, we have

$$\left.\begin{aligned} &d\Delta\frac{dZ}{d\xi'}\delta\xi + \Delta\frac{dZ}{d\xi'}d\delta\xi + (\ ) \\ &-d\delta\frac{dZ}{d\xi'}\Delta\xi - \delta\frac{dZ}{d\xi'}d\Delta\xi - (\ ) \end{aligned}\right\} = 0.$$

This equation is integrable, and gives

$$\Delta\frac{dZ}{d\xi'} - \delta\xi\,\delta\frac{dZ}{d\xi'}\Delta\xi + (\ ) = C.$$

Now since ξ', ψ', ϕ', do not enter into V, $\frac{dZ}{d\xi'} = \frac{dT}{d\xi'}$. Making this substitution, and supplying the deficient terms, we arrive at the theorem sought,

$$\left.\begin{aligned} &\Delta\frac{dT}{d\xi'}\delta\xi + \Delta\frac{dT}{d\psi'}\delta\psi + \Delta\frac{dT}{d\phi'}\delta\phi \\ &-\delta\frac{dT}{d\xi'}\Delta\xi - \delta\frac{dT}{d\psi'}\Delta\psi - \delta\frac{dT}{d\phi'}\Delta\phi \end{aligned}\right\} = C.$$

The investigation of this theorem in the *Mec. Analytique* occupies nearly four pages, owing to the lengthened developments which are there introduced.

I shall now proceed to demonstrate from the general transformed equation of motion, the principles of the conservation of living forces, and of least action. The former of these has been thence deduced by Lagrange. I am not however aware that the latter has been obtained from the same equation, either by the discoverer of the Calculus of Variations, or by any subsequent author.

The general differential equation being written under the form

$$\left(\frac{d}{dt}\frac{d\mathrm{T}}{d\xi'}-\frac{d\mathrm{T}}{d\xi}\right)\delta\xi+\left(\frac{d}{dt}\frac{d\mathrm{T}}{d\psi'}-\frac{d\mathrm{T}}{d\psi}\right)\delta\psi+\left(\frac{d}{dt}\frac{d\mathrm{T}}{d\phi'}-\frac{d\mathrm{T}}{d\phi}\right)\delta\phi+\delta\mathrm{V}=0\,(\mathrm{A}),$$

we have, on changing δ into d,

$$\left(\frac{d}{dt}\frac{d\mathrm{T}}{d\xi'}-\frac{d\mathrm{T}}{d\xi}\right)d\xi+(\ )+d\mathrm{V}=0\ldots\ldots\ldots(\mathrm{B}).$$

Integrating, we obtain

$$\frac{d\mathrm{T}}{d\xi'}\xi+\frac{d\mathrm{T}}{d\psi'}\psi'+\frac{d\mathrm{T}}{d\phi'}\phi'-\mathrm{T}+\mathrm{V}=\mathrm{H}\ (\text{a constant})\ldots(\mathrm{C});$$

which, by the theorem for homogeneous functions, becomes

$$\mathrm{T}+\mathrm{V}=\mathrm{H},$$

and expounds the principle of the conservation of living forces. Take now the variation of (C),

$$\left.\begin{aligned}&\delta\frac{d\mathrm{T}}{d\xi'}\xi'+\frac{d\mathrm{T}}{d\xi'}\delta\xi'+(\ )\\&-\frac{d\mathrm{T}}{d\xi}\delta\xi-\frac{d\mathrm{T}}{d\xi'}\delta\xi'-(\ )\end{aligned}\right\}+\delta\mathrm{V}=0.$$

The even terms mutually destroying each other, we have, on equating the remaining ones with the first side of (A),

$$\delta\frac{d\mathrm{T}}{d\xi'}\xi'-\frac{d\mathrm{T}}{d\xi}\delta\xi+(\ )+\delta\mathrm{V}=\frac{d}{dt}\frac{d\mathrm{T}}{d\xi'}\delta\xi-\frac{d\mathrm{T}}{d\xi}\delta\xi+(\ )+\delta\mathrm{V};$$

$$\text{therefore}\quad\delta\frac{d\mathrm{T}}{d\xi'}\xi'+(\ )=\frac{d}{dt}\frac{d\mathrm{T}}{d\xi'}\delta\xi+(\ ).$$

To this add the identical equation,

$$\frac{d\mathrm{T}}{d\xi'}\delta\xi'=\frac{d\mathrm{T}}{d\xi'}\frac{d}{dt}\delta\xi;$$

and observing that each side becomes integrable, we have, on performing the integrations and supplying the deficient terms,

$$\delta\left(\frac{d\mathrm{T}}{d\xi'}\xi'+\frac{d\mathrm{T}}{d\psi'}\psi'+\frac{d\mathrm{T}}{d\phi'}\phi'\right)=\frac{d}{dt}\left(\frac{d\mathrm{T}}{d\xi'}\delta\xi+\frac{d\mathrm{T}}{d\psi'}\delta\psi+\frac{d\mathrm{T}}{d\phi'}\delta\phi\right).$$

The first side of the above equation is, by the theorem for homogeneous functions of the second degree, equal to $\delta(2\mathrm{T})$.

Hence, if $\delta\xi$, $\delta\psi$, $\delta\phi$, are supposed to vanish for the extreme points, we have

$$\delta \int 2\mathrm{T}\,dt = 0,$$

the same result as was obtained in the former investigation, from the equation in rectangular co-ordinates.

II.—ON AN EXPRESSION FOR ANY POSITIVE INTEGRAL POWER OF A LOGARITHM.

THE peculiar method by which Lagrange, in the *Théorie des Fonctions*, has arrived at the expansion of $\log_a x$, we propose to apply to the determination of an expression for $(\log_a x)^{m+1}$, where m is any positive integer.

Assume $a^y = x$:

then clearly

$$\{1 + (a-1)\}^y = 1 + (x-1),$$

and therefore

$$\{1 + (a-1)\}^{ny} = \{1 + (x-1)\}^n,$$

where n is any quantity whatever.

Hence, by the binomial theorem,

$$1 + ny(a-1) + \frac{ny\,(ny-1)}{1.2}(a-1)^2 + \frac{ny(ny-1)\,(ny-2)}{1.2.3}(a-1)^3 + \ldots$$

$$= 1 + n(x-1) + \frac{n(n-1)}{1.2}(x-1)^2 + \frac{n(n-1)\,(n-2)}{1.2.3}(x-1)^3 + \ldots$$

Suppose now that

$^{m+1}\mathrm{S}_1$ = the sum of the numbers 1, 2, 3, ... $m+1$,

$^{m+2}\mathrm{S}_2$ = the sum of the products of every two of 1, 2, 3,...$m+2$,

$^{m+3}\mathrm{S}_3$ = ... three of 1, 2, 3,...$m+3$,

&c. = &c.

Then clearly the coefficient of n^{m+1}, if m be a positive integer, will, in the former member of the equation, be

$$y^{m+1}.\left\{\frac{(a-1)^{m+1}}{1.2.3..(m+1)} - {}^{m+1}\mathrm{S}_1.\frac{(a-1)^{m+2}}{1.2.3..(m+2)} + {}^{m+2}\mathrm{S}_2.\frac{(a-1)^{m+3}}{1.2.3..(m+3)} - \ldots\right\}$$

and in the latter

$$\frac{(x-1)^{m+1}}{1.2.3\ldots(m+1)} - {}^{m+1}\mathrm{S}_1.\frac{(x-1)^{m+2}}{1.2.3\ldots(m+2)} + {}^{m+2}\mathrm{S}_2.\frac{(x-1)^{m+3}}{1.2.3\ldots(m+3)} - \ldots$$

and therefore, equating these two expressions, which, n being a

perfectly arbitrary quantity, we are at liberty to do by the theory of indeterminate coefficients, we have

$$y^{m+1} \text{ or } (\log_a x)^{m+1} =$$

$$\frac{\dfrac{(x-1)^{m+1}}{1.2.3\ldots(m+1)} - {}^{m+1}S_1 \cdot \dfrac{(x-1)^{m+2}}{1.2.3\ldots(m+2)} + {}^{m+2}S_2 \cdot \dfrac{(x-1)^{m+3}}{1.2.3\ldots(m+3)} - \ldots}{\dfrac{(a-1)^{m+1}}{1.2.3\ldots(m+1)} - {}^{m+1}S_1 \cdot \dfrac{(a-1)^{m+2}}{1.2.3\ldots(m+2)} + {}^{m+2}S_2 \cdot \dfrac{(a-1)^{m+3}}{1.2.3\ldots(m+3)} - \ldots}$$

If we put $m = 0$, this formula is evidently reduced to

$$\log_a x = \frac{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \ldots},$$

which is the ordinary expression for $\log_a x$.

W. W.

III.—ON THE GENERAL INTERPRETATION OF EQUATIONS BETWEEN TWO VARIABLES IN ALGEBRAIC GEOMETRY.

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1. The object of the present paper is to discuss the general geometrical signification of an equation involving two variables, x and y. The propositions which we propose to establish are the two following:—first, that when both the magnitudes and the affections of x and y experience simultaneously every conceivable variation consistently with their mutual relation, the equation will represent a curve surface; and, secondly, that when unlimited variations of magnitude are assigned to x and y, mutually consistent, while the affection of either of them is restricted to a constant state, the equation will represent a curve line, there being a different curve line for each different constant state of the affection of the restricted variable, and that every one of this series of curve lines lies in the surface of our first proposition.

In the demonstration of these two propositions, we have in the first instance employed the symbol of revolution $+^p$ for the construction of the indefinite number of pairs of axes of x and y, in a manner different from that which has been adopted by Mr. Gregory in his article on the Existence of Branches of Curves in several Planes, (see Vol. I. No. VI. of this Journal), the only principle by which we consider that we are necessarily to be guided in this matter being, that supposing $+^r a$ and $+^s \beta$ to be corresponding values of x and y, the corresponding conjugate axes of x and y shall make with the original axes the angles $2r\pi$ and $2s\pi$ respectively, their position being additionally and finally defined by any

fixed law to which we may choose to subject them. We have also, in the second instance, shewn that, according either to the method developed by Mr. Gregory, or to any other admissible method of constructing the pairs of conjugate axes, the same two propositions are equally true.

In the investigations upon which we are about to enter, we shall explain the method of translating the expressions both for the curves and for the surface, from the affectional equation between the two variables x and y, to equivalent quantitive equations between three variables, x, y, and z.

2. Let the general equation between two variables, x and y, be represented by

$$f(x, y, A_1, A_2, A_3, \ldots A_\tau) = 0 \ldots\ldots\ldots (1),$$

where f is a general symbol of functionality, and A_1, A_2, A_3, $\ldots A_\tau$ are the arbitrary constants of the equation.

Let $A_1 = \phi_1(+, -) . k_1$,
$A_2 = \phi_2(+, -) . k_2$,
&c. = &c.
$A_\tau = \phi_\tau(+, -) . k_\tau$,

where k_1, k_2, k_3, $\ldots\ldots k_\tau$ are merely symbols of quantity, and $\phi_1(+, -)$, $\phi_2(+, -)$, $\ldots \phi_\tau(+, -)$ any functions whatever of $+$ and $-$. These it is evident are the most general forms for the arbitrary constants of which we can form any conception. Now we know that

$$+ = \cos 2\pi + (-)^{\frac{1}{2}} \sin 2\pi,$$

$$\text{and} \quad - = \cos \pi + (-)^{\frac{1}{2}} \sin \pi,$$

and therefore

$$\phi_1(+, -) = \phi_1 \{\cos 2\pi + (-)^{\frac{1}{2}} \sin 2\pi, \cos \pi + (-)^{\frac{1}{2}} \sin \pi\},$$

and similarly for the expressions for A_2, A_3, $\ldots A_\tau$.

But it is clear that in all cases the expression for $\phi_1(+, -)$ can be expanded by some algebraical process, so as to give a result of the form

$$M + (-)^{\frac{1}{2}} N,$$

where M and N are symbols of quantity.

Hence we have $A_1 = \phi_1(+, -) . k_1$,
$= Mk_1 + (-)^{\frac{1}{2}} Nk_1$.

Put $Mk_1 = a_1 \cos 2m_1\pi$ and $Nk_1 = a_1 \sin 2m_1\pi$, and we have

$$A_1 = a_1 \{\cos 2m_1\pi + (-)^{\frac{1}{2}} \sin 2m_1\pi\},$$
$$= \{\cos 2\pi + (-)^{\frac{1}{2}} \sin 2\pi\}^{m_1} . a_1,$$
$$= +^{m_1} a_1.$$

In the same way we may shew, that

$$\begin{aligned} A_2 &= +^{m_2} . a_2, \\ A_3 &= +^{m_3} . a_3, \\ \&c. &= \&c. \\ A_\tau &= +^{m_\tau} . a_\tau. \end{aligned}$$

The equation (1) then, when the affections of its parameters are made explicit, becomes

$$f\{x, y, +^{m_1}a_1, +^{m_2}a_2, +^{m_3}a_3, \ldots +^{m_\tau}a_\tau\} = 0.$$

Again, the general expression for the variable value of x is, as we may shew by analogous reasoning, $+^r a$, where r and a are variable quantities, whose variations are mutually independent, and where we suppose a to be a symbol of magnitude alone, $+^r$ being the representative of the affection of position. Let $+^s\beta$ be the corresponding value for y.

Hence, the equation (1) is equivalent to

$$f\{+^r a, +^s\beta, +^{m_1}a_1, +^{m_2}a_2, +^{m_3}a_3, \ldots\ldots +^{m_\tau}a_\tau\} = 0;$$

and if we put

$$\begin{aligned} +^r &= \cos 2r\pi + (-)^{\frac{1}{2}} \sin 2r\pi, \\ +^s &= \cos 2s\pi + (-)^{\frac{1}{2}} \sin 2s\pi, \\ +^{m_1} &= \cos 2m_1\pi + (-)^{\frac{1}{2}} \sin 2m_1\pi, \\ \&c. &= \&c \end{aligned}$$

and then expand by some algebraical process the former member of this equation, we shall obviously get an equation of the form

$$\phi(a, \beta, r, s) + (-)^{\frac{1}{2}} \psi(a, \beta, r, s) = 0,$$

where ϕ and ψ are symbols of functionality; and this equation clearly resolves itself into the two following:

$$\phi(a, \beta, r, s) = 0 \ldots\ldots\ldots\ldots (2),$$

and $\psi(a, \beta, r, s) = 0 \ldots\ldots\ldots\ldots (3).$

3. Let Ox, Oy, be the axes of x and y, and from O draw Oz perpendicular to the plane of x, y. In the plane xOz draw OE equal to a, and making an angle $2r\pi$ with the axis of x. And in the plane yOz draw OF equal to β, and making an angle $2s\pi$ with the axis of y. From E draw EP parallel to OF, and meeting FP drawn from F parallel to OE in the point P. Then is the point P a point in the locus of the equation (1).

From E and F draw EM and FN, meeting respectively in M and N, the axes of x and y at right angles, and from P let fall PQ meeting at right angles in the point Q the plane of x, y. Also from E draw ER parallel to MQ, and meeting PQ in R. Then clearly RQ is equal to EM; and since EP, ER, are evidently equal

respectively to OF and ON and the angle PER to the angle FON, it is clear that PR is equal to FN.

Let OM, ON, PQ be called respectively x', y', z'. Then clearly, from what has been said, we have

$$x' = a \cos 2r\pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$y' = \beta \cos 2s\pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$\text{and } z' = a \sin 2r\pi + \beta \sin 2s\pi \ldots\ldots\ldots (6).$$

4. If between the five equations (2), (3), (4), (5), (6), we eliminate the four quantities a, β, r, s, we shall evidently obtain an equation

$$\theta(x', y', z') = 0 \ldots\ldots\ldots\ldots (7),$$

where $\theta(x', y', z')$ represents some function of x', y', z'.

Thus we see that the equation (1), when interpreted according to the utmost generality of the meanings of x and y, denotes a surface represented quantitively by the equation (7).

5. If instead of supposing r a variable quantity, we consider it to be constant, we shall get as the result of eliminating a, β, and s between the five equations (2), (3), (4), (5), (6), the two following equations:

$$\chi(x', y', z') = 0 \ldots\ldots\ldots\ldots (8)$$

$$\omega(x', y', z') = 0 \ldots\ldots\ldots\ldots (9),$$

where $\chi(x', y', z')$, $\omega(x', y', z')$, represent certain functions of x', y', z'. Thus we see that if the affection of x in the equation (1) remain invariable, this equation will represent a curve line whose equations in x', y', z' are (8) and (9).

6. Since from what has been said, it is evident that the equation (7) must result from the elimination of r between the equations (8) and (9), it is clear that the equation (7) must correspond to a surface formed by a series of curves represented by the equations (8) and (9) when r experiences every degree of variation. Hence, conversely, every curve of the class represented by the equations (8) and (9) must lie altogether within the surface represented by the equation (7).

7. In illustration of the principles which we have developed above, we will furnish one or two examples. Let us take as a first example the equation

$$x^2 + y^2 = A^2.$$

Assuming

$$x = +\,^r a,\ y = +\,^s \beta,\ k = +\,^m b,$$

we have

$$+\,^{2r} a^2 + +\,^{2s} \beta^2 = +\,^{2m} b^2.$$

Hence putting for $+$ its trigonometrical value, and equating the coefficient of $(-)^{\frac{1}{2}}$ and the part which is independent of this symbol separately to zero, we have

$$a^2 . \cos 4r\pi + \beta^2 . \cos 4s\pi = b^2 . \cos 4m\pi,$$

$$\text{and } a^2 . \sin 4r\pi + \beta^2 . \sin 4s\pi = b^2 . \sin 4m\pi.$$

From these two equations and the equations (4), (5), (6), we have, by the elimination of α and β, the following equations:

$$x'^2 \cdot \frac{\cos 4r\pi}{\cos^2 2r\pi} + y'^2 \cdot \frac{\cos 4s\pi}{\cos^2 2s\pi} = b^2 \cdot \cos 4m\pi$$

$$x'^2 \cdot \frac{\sin 4r\pi}{\cos^2 2r\pi} + y'^2 \cdot \frac{\sin 4s\pi}{\cos^2 2s\pi} = b^2 \cdot \sin 4m\pi$$

$$z' = x' \cdot \tan 2r\pi + y' \cdot \tan 2s\pi.$$

Now the two former of these equations may be written

$$x'^2 + y'^2 - x'^2 \tan^2 2r\pi - y'^2 \tan^2 2s\pi = b^2 \cdot \cos 4m\pi$$

$$2x'^2 \cdot \tan 2r\pi + 2y'^2 \cdot \tan 2s\pi = b^2 \cdot \sin 4m\pi;$$

and, therefore, by virtue of the third of them, we have the two following equations:

$$x'^2 + y'^2 - x'^2 \tan^2 2r\pi - (z' - x' \tan 2r\pi)^2 = b^2 \cdot \cos 4m\pi,$$

and

$$2x'(x' - y') \tan 2r\pi + 2y'z' = b^2 \cdot \sin 4m\pi.$$

If r be considered of invariable magnitude, these two equations represent a curve line. But if r be reckoned variable, we have, eliminating it between them, the following result:

$$4(x' - y')^2(x'^2 + y'^2) = 4b^2 \cdot (x' - y')^2 \cdot \cos 4m\pi + (2x'z' - b^2 \sin 4m\pi)^2 + (2y'z' - b^2 \sin 4m\pi)^2,$$

which is the equation to the surface represented by the affectional equation

$$x^2 + y^2 = A^2.$$

8. As a second example we will take the equation

$$y^2 = Ax.$$

Assuming $x = +\,^r a$, $y = +\,^s \beta$, $A = +\,^m b$, we have

$$+\,^{2s}\beta^2 = +\,^{r+m} ba,$$

whence

$$\{\cos 4s\pi + (-)^{\frac{1}{2}} \sin 4s\pi\} \cdot \beta^2 = \{\cos 2(r+m)\pi + (-)^{\frac{1}{2}} \sin 2(r+m)\pi\}\, ba;$$

and, therefore,

$$\beta^2 \cos 4s\pi = ba \cos 2(r+m)\pi,$$

$$\text{and } \beta^2 \sin 4s\pi = ba \sin 2(r+m)\pi.$$

Hence we have

$$\beta^2 = ba,$$

$$\text{and } 2s\pi = (r+m)\pi;$$

and therefore by the equations (4) and (5), we have

$$bx' = \beta^2 \cos 2r\pi$$

$$y'^2 = \beta^2 \cos^2 2s\pi;$$

and, therefore,

$$bx' \cos^2 2s\pi = y'^2 . \cos^2 2r\pi ;$$

and, therefore,

$$bx' \cos^2 (r+m)\pi = y'^2 . \cos^2 2r\pi ;$$

and from the equations (4), (5), (6),

$$z' = x' \tan 2r\pi + y' \tan 2s\pi,$$

$$\text{or } z' = x' \tan 2r\pi + y' \tan (r+m)\pi.$$

If r be invariable, then these two last equations represent a curve line. And supposing r to be variable, we shall obtain for the locus of the equation $y^2 = Ax$ the appropriate equation in x', y', z', by eliminating the r between the two equations to the curve line.

If we put $r = 0$, and $m = 0$, we get for the equations to the curve

$$bx' = y'^2$$
$$z' = 0,$$

which are the equations to the common parabola in the plane x', y'.

9. As a third example we will take the equation of the first degree in x and y.

$$Ax + By = 1 \ldots\ldots \ldots\ldots (A).$$

Assume $A = + {}^m a$, $B = + {}^n b$, $x = + {}^r \alpha$, and $y = + {}^s \beta$, and this equation assumes the form

$$+ {}^{m+r} a\alpha + + {}^{n+s} b\beta = 1.$$

Hence, putting

$$+ = \cos 2\pi + (-)^{\frac{1}{2}} \sin 2\pi,$$

we have

$$a\alpha \cos.2(m+r)\pi + b\beta \cos . 2(n+s)\pi + (-)^{\frac{1}{2}} \{a\alpha \sin . 2(m+r)\pi + b\beta \sin . 2(n+s)\pi\} = 1,$$

and this equation resolves itself into the two following:

$$a\alpha \cos . 2(m+r)\pi + b\beta \cos . 2(n+s)\pi = 1 \ldots\ldots (B)$$

$$\text{and } a\alpha \sin . 2(m+r)\pi + b\beta \sin . 2(n+s)\pi = 0 \ldots\ldots (C).$$

Multiplying the former and the latter of these equations respectively by $\cos 2n\pi$ and $\sin 2n\pi$, and adding the results, we get

$$a\alpha \cos . 2(m+r-n)\pi + b\beta \cos 2s\pi = \cos 2n\pi \ldots\ldots (D),$$

and multiplying them by $\cos 2m\pi$ and $\sin 2m\pi$ respectively, and adding the results, we have

$$a\alpha \cos 2r\pi + b\beta \cos 2(n+s-m)\pi = \cos 2m\pi \ldots\ldots (E).$$

Again, multiplying the equations (4), (5), (6), respectively by the three following expressions:

$$a\{b \cos 2(n-m)\pi - a\}, \; b\{b - a \cos 2(n-m)\pi\} y', \; ab \sin 2(n-m)\pi,$$

and adding the results, we have for the left-hand member of the resulting equation

$$a\{b\cos 2(n-m)\pi - a\}x' + b\{b - a\cos 2(n-m)\pi\}y' + ab\sin 2(n-m)\pi \cdot z',$$

and for the right-hand member

$$\{ab\cos 2(n-m)\pi - a^2\}\alpha\cos 2r\pi + \{b^2 - ab\cos 2(n-m)\pi\}\beta\cos 2s\pi + aba\sin 2(n-m)\pi \cdot \sin 2r\pi + ab\beta\sin 2(n-m)\pi \cdot \sin 2s\pi,$$

or

$$b\{a\alpha\cos 2(m+r-n)\pi + b\beta\cos 2s\pi\} - a\{a\alpha\cos 2r\pi + b\beta\cos 2(n+s-m)\pi\};$$

and, therefore, by virtue of the equations (D) and (E),

$$b\cos 2n\pi - a\cos 2m\pi.$$

Hence the affectional equation (A) is equivalent to the following quantitive one,

$$a\{b\cos 2(n-m)\pi - a\}x' + b\{b - a\cos 2(n-m)\pi\}y' + ab\sin 2(n-m)\pi \cdot z' = b\cos 2n\pi - a\cos 2m\pi;$$

or, if symmetry of expression be considered necessary,

$$2a\{b\cos 2(m-n)\pi - a\}x' + ab\sin 2(n-m)\pi \cdot z' + 2a\cos 2m\pi = 2b\{a\cos 2(n-m)\pi - b\}y' + ba\sin 2(m-n)\pi \cdot z' + 2b\cos 2n\pi \ldots\ldots (F),$$

which is an equation of the first degree in x', y', z'.

Hence we see that the equation of the first degree between two variables represents a plane surface, when the utmost generality of signification and of variation is assigned to its elemental letters.

10. Let the equation to the plane surface which is represented by the equation (F) be conceived to be expressed under the form

$$\frac{x'}{\beta_1} + \frac{y'}{\beta_2} + \frac{z'}{\beta_3} = 1 \ldots\ldots\ldots\ldots (G).$$

where β_1, β_2, β_3, are the parameters of the equation.

Then comparing this equation with the equation (F), we have the three following relations:

$$a\beta_1\{b\cos 2(n-m)\pi - a\} = b\cos 2n\pi - a\cos 2m\pi \ldots\ldots (H)$$

$$b\beta_2\{b - a\cos 2(n-m)\pi\} = b\cos 2n\pi - a\cos 2m\pi \ldots\ldots (I)$$

$$\text{and } ab\beta_3\sin 2(n-m)\pi = b\cos 2n\pi - a\cos 2m\pi \ldots\ldots (K).$$

Now it is evidently sufficient and necessary for the conservation of the identity of the plane (G), and therefore of the plane (F), with which it is coincident, that the values of the quantities β_1, β_2, β_3, experience separately no variation. Hence, clearly, so long as the three relations (H), (I), (K), are satisfied for constant values of

these quantities, the identity of the locus of the affectional equation (A) will be secured. But it is obvious that since the three equations (H), (I), (K), involve four quantities a, b, m, n, the values of these quantities must remain individually indeterminate. Hence we see that it is not necessary, as we have been supposing, that the quantities a, b, m, n, should remain constant in order to secure the identity of the locus of the equation (A), but merely that they preserve always the mutual relations which we have explained.

11. If between the five equations (D), (E), (H), (I), (K), we eliminate successively a, b, s, n, and a, b, β, n, we shall evidently get respectively

$$\beta = \phi(a, r, m)$$
$$\text{and } s = \chi(a, r, m),$$

where $\phi(a, r, m)$ represents a function of a, r, m, not involving any of the quantities a, b, s, n, and $\chi(a, r, m)$ a function of a, r, m, not involving any of the quantities a, b, β, n.

From these expressions for β and s, it appears that for any assigned values of a and r, β and s will experience an infinite variety of values with the variation of the value of m; and by what has been said in the preceding section, it is clear that, without affecting the identity of the locus of the equation (A), we may assign to m whatever values we please. This amounts to saying, that having taken a certain angle EOx, and a certain length OE, the angle FOy and the length OF will not be defined in magnitude, but that without affecting the identity of the plane surface represented by the equation (A), an infinite variety of points P in it will be determinable for any assigned magnitudes of the angle EOx, and the length OE, according to the value which we assign to m. Hence we see that the equation (A) not only represents a definite plane, but likewise furnishes us with an infinite number of ways of arriving geometrically at each of its constituent points.

12. If we wish to transform the equation to any plane from the quantitive shape (G) to the affectional shape (A), it is plain that from the three equations (H), (I), (K), we must obtain for a, b, and n, expressions in terms of β_1, β_2, β_3, and m, and we shall have for the resulting equation a form

$$+ {}^{m}\phi(\beta_1, \beta_2, \beta_3, m) \cdot x + + {}^{\psi(\beta_1, \beta_2, \beta_3, m)} \cdot \chi(\beta_1, \beta_2, \beta_3, m) \cdot y = 1,$$

the locus of which will be the same whatever value we may assign to m.

13. If between the three equations (H), (I), (K), we eliminate a and b, we shall ultimately arrive at the expression

$$(\beta_1 - \beta_2)\beta_3 = (\beta_1\beta_2 + \beta_3^2) \tan 2(n - m)\pi,$$

a result which shews that the value of $n - m$ must always remain constant for the conservation of the identity of the locus of the equation (A).

We may also easily arrive at the following relation from the same three equations

$$\frac{b}{a} - \frac{a}{b} = \frac{(\beta_1+\beta_2)\beta_3}{\beta_1\beta_2} \sin 2(n-m)\pi,$$

which, in conjunction with the previous result, shews that the relation between a and b must be invariable.

14. Again, suppose that r is constant, or that the affections of x in the equation (A) are restricted.

Then multiplying the equation (D) by $\cos 2r\pi$, and introducing the relations afforded by the equations (4) and (5), we have

$$ax' \cos 2(m-n+r)\pi + by' \cos 2r\pi = \cos 2n\pi \cdot \cos 2r\pi \ldots (a).$$

and multiplying the equation (E) by $\cos 2s\pi$, we have in like manner

$$ax' \cos 2s\pi + by' \cos 2(n-m+s)\pi = \cos 2m\pi \cdot \cos 2s\pi \ldots (b),$$

and from the three equations (4), (5), (6), we have

$$z' = x' \tan 2r\pi + y' \tan 2s\pi,$$

and between this last equation and the equation (b), we can readily arrive at the equation

$$\{a - b \tan 2r\pi \cdot \sin 2(m-n)\pi\} \cdot x' + b \cos 2(m-n)\pi \cdot y' + bz' \sin 2(m-n)\pi = \cos 2m\pi \ldots\ldots (c).$$

Hence we see that when x is restricted in point of its affections, the equation (A) represents a straight line whose equations are (a) and (c) expressed quantitively.

15. If m, n, r, be all put equal to zero, the equations (B) and (C) become

$$\begin{aligned} a\alpha + b\beta \cos 2s\pi &= 1 \\ b\beta \sin 2s\pi &= 0, \end{aligned}$$

whence $s = 0$, and therefore,

$$a\alpha + b\beta = 1.$$

Hence from (4), (5), (6), we have

$$x' = \alpha, \quad y' = \beta, \quad z' = 0,$$

and, therefore, we get

$$\begin{aligned} ax' + by' &= 1 \\ z' &= 0, \end{aligned}$$

as the equations to the straight line represented by the equation (A).

16. We will take a single example of the case in which the values of x in the equation (A) are restricted to mere magnitude.

Let the equation be

$$y = +\tfrac{1}{4} k (x-l).$$

Put the equation

$$+^{m} ax + +^{n} by = 1,$$

under the form

$$y = - +^{m-n} \frac{a}{b}\left(x - +^{-m} \frac{1}{a}\right),$$

and comparing the coefficients of like terms in the two, we have

$$l = +^{-m} \frac{1}{a}, \text{ whence } m = 0 \text{ and } a = \frac{1}{l},$$

$$\text{and } +^{\frac{1}{4}} k = - +^{m-n} \frac{a}{b},$$

$$\text{whence } n = -\frac{1}{4} \text{ and } b = -\frac{1}{lk}.$$

Hence, by substituting these values in the equations (a) and (c) and putting $r = 0$, we have

$$y' = 0,$$

$$\text{and } ax' + bz' = 1,$$

$$\text{or } \frac{1}{l} x' - \frac{1}{kl} z' = 1,$$

$$\text{or } z' = k(x' - l),$$

which are the two equations to the iine.

Thus we see that prefixing the symbol $+^{\frac{1}{4}}$ to k in the equation

$$y = k(x - l),$$

is equivalent to turning the corresponding line into a plane at right angles to that of x, y, without altering its inclination to the axis of x, or the point in which it cuts it.

17. We will now proceed to shew that the two propositions, which it has been the object of the present paper to establish, hold true likewise, when, instead of the application which we have chosen of the symbol of revolution $+^{p}$, we adopt the application developed in Mr. Gregory's article.

In fact, suppose as before that

$$x = +^{r} \alpha \text{ and } y = +^{s} \beta.$$

Transfer the axis Ox through an angle $2r\pi$ in a plane perpendicular to Oy into a new axis Ox_1, and then turn Oy through an angle $2s\pi$ in a plane perpendicular to the axis Ox_1 into an axis Oy_1. Measure a length α along Ox_1, and a length β along Oy_1; and then in the new system of axes Ox_1 and Oy_1, consider α and β as the co-ordinates of the point defined by the equations $x = +^{r} \alpha$ and $y = +^{s} \beta$.

Let x', y', z', represent the quantitive co-ordinates of this same point. Then, by Euler's formulæ for transformation from one system of rectangular co-ordinates to another likewise rectangular, we

shall have, recollecting that in the new system the co-ordinate perpendicular to the plane of x_1Oy_1 is put equal to zero,

$$x' = \alpha \cos 2r\pi - \beta \sin 2r\pi \,.\, \sin 2s\pi,$$
$$y' = \beta \cos 2s\pi,$$
$$z' = \alpha \sin 2r\pi + \beta \cos 2r\pi \,.\, \sin 2s\pi;$$

and dealing with these three equations in the place of the equations (4), (5), (6), we shall, by exactly the same kind of reasoning as we have employed in the investigations of the preceding sections, be enabled to determine the curve lines and the curve surface belonging to the equation $f(x, y) = 0$, and to shew that the curve lines all lie within the curve surface.

And, generally, if we were to select any other admissible law for the determination of the indefinite number of pairs of conjugate axes, and that too whether the prime axes be rectangular or oblique, we should evidently obtain for x', y', z', expressions functional of α, β, r, s, and analogous conclusions in respect to the curve lines and the curve surface would obviously be established.

IV.—VARIATION OF NODE AND INCLINATION.

THE following method of finding the variations of the inclination and longitude of the node, is more convenient than that given in Pratt's *Mechanical Philosophy*, p. 336.

Adopting the notation usual in the lunar theory, we have

$$s = k \sin(\theta - \gamma) \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{also } \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{d\mathrm{R}}{dz} = 0 \ldots\ldots\ldots\ldots (2).$$

In the disturbed orbit, (1) and its first derived equation will be true, as if the elements were invariable; which gives the equation

$$\sin(\theta - \gamma)\frac{dk}{dt} - k\cos(\theta - \gamma)\frac{d\gamma}{dt} = 0 \ldots\ldots (3),$$

and differentiating (1) a second time, there is

$$\frac{d^2s}{dt^2} = k\frac{d\theta}{dt}\cdot\frac{d}{d\theta}\left(\cos(\theta - \gamma)\frac{d\theta}{dt}\right) + \left(\cos(\theta - \gamma)\frac{dk}{dt} + k\sin(\theta - \gamma)\frac{d\gamma}{dt}\right)\frac{d\theta}{dt}.$$

The second and third terms are those due to perturbation. Also, the inclination being very small, the effect of perturbation on $\frac{d^2z}{dt^2}$, or which is the same thing, on $\frac{d^2.\rho s}{dt^2}$, will be sensible only in

the term $\rho \frac{d^2 s}{dt^2}$. Hence, equating the perturbation and its effect, we have

$$\rho \frac{d\theta}{dt}\left(\cos(\theta-\gamma)\frac{dk}{dt} + k\sin(\theta-\gamma)\frac{d\gamma}{dt}\right) + \frac{d\mathrm{R}}{dz} = 0 \ldots (4),$$

and eliminating in turn $\frac{dk}{dt}$, $\frac{d\gamma}{dt}$, by (3), we get

$$\rho \frac{d\theta}{dt} k \frac{d\gamma}{dt} + \frac{d\mathrm{R}}{dz}\sin(\theta-\gamma) = 0,$$

$$\rho \frac{d\theta}{dt}\frac{dk}{dt} + \frac{d\mathrm{R}}{dz}\cos(\theta-\gamma) = 0.$$

Again, the inclination being small,

$$\frac{d\mathrm{R}}{dk} = \frac{d\mathrm{R}}{dz}\cdot\frac{dz}{dk} = \rho\frac{d\mathrm{R}}{dz}\cdot\frac{ds}{dk} = \rho\frac{d\mathrm{R}}{dz}\sin(\theta-\gamma) \text{ nearly,}$$

$$\text{and } \frac{d\mathrm{R}}{d\gamma} = \frac{d\mathrm{R}}{dz}\cdot\frac{dz}{d\gamma} = \rho\frac{d\mathrm{R}}{dz}\cdot\frac{ds}{d\gamma} = -\rho k\frac{d\mathrm{R}}{dz}\cos(\theta-\gamma);$$

$$\therefore \rho^2\frac{d\theta}{dt}k\frac{d\gamma}{dt} = -\frac{d\mathrm{R}}{dk},$$

$$\text{and } \rho^2\frac{d\theta}{dt}k\frac{dk}{dt} = \frac{d\mathrm{R}}{d\gamma},$$

$$\rho^2\frac{d\theta}{dt} = h = \sqrt{\mu a(1-e^2)} = \frac{\mu\sqrt{1-e^2}}{na};$$

$$\therefore \frac{d\gamma}{dt} = -\frac{1}{k}\cdot\frac{na}{\mu\sqrt{1-e^2}}\frac{d\mathrm{R}}{dk} \ldots\ldots\ldots (5),$$

$$\frac{dk}{dt} = \frac{1}{k}\frac{na}{\mu\sqrt{1-e^2}}\frac{d\mathrm{R}}{d\gamma} \ldots\ldots\ldots\ldots (6),$$

which agree with the known results, k being $= \tan \mathrm{I}$, or $\sin \mathrm{I}$, *quam proximè*, and γ being what Mr. Pratt denotes by Ω. (The squares, &c. of I are neglected throughout.)

R. L. E.

V.—ON THE INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

[By G. Boole.]

In an article in the first number of this Journal, (Vol. I. p. 22.) Mr. Gregory has applied the method of the separation of symbols to the Integration of Linear Differential Equations with Constant

Coefficients. The greater part of the process is at once simple and direct, which gives this method an advantage over that of the Variation of Parameters; but the reduction of the complex inverse operations to a sum of similar simple terms by means of integration by parts, is laborious and tedious, and may be very greatly abbreviated by a method which I propose here to exhibit.

If we represent the general equation

$$\left(\frac{d^n}{dx^n} + A_1\frac{d^{n-1}}{dx^{n-1}} + A_2\frac{d^{n-2}}{dx^{n-2}} + \&c. + A_n\right) y = X \ldots (1),$$

$$\text{by } f\left(\frac{d}{dx}\right) y = X,$$

we deduce, as is done in the paper referred to,

$$y = \left\{f\left(\frac{d}{dx}\right)\right\}^{-1} X \ldots\ldots\ldots (2).$$

Now, instead of splitting the operating factors into simple binomial factors, and operating with them in succession, which renders it necessary to simplify the result by integration by parts, we may at once resolve the general operating factors into the sum of a number of simple binomial factors, exactly as in ordinary Algebra a rational fraction is decomposed into the sum of a number of simple fractions. The expression

$$\left\{f\left(\frac{d}{dx}\right)\right\}^{-1},$$

is the same in form as the rational fraction

$$\{f(z)\}^{-1} = \frac{1}{z^n + A_1 z^{n-1} + A_2 z^{n-2} + \&c. + A_n}.$$

Now the method of the resolution of this into a sum of partial fractions, is independent of any properties of the variable, except the three which have been shown by Mr. Gregory (Vol. I. p. 31.) to be common to the symbol $\frac{d}{dx}$, and to the algebraical symbols generally supposed to represent numbers. Consequently the same means which enable us to determine the form of the partial fractions in ordinary Algebra, may be applied to the circumstances of the case now under consideration. This, it will be seen, is nothing more than a farther extension of the application of the principles on which the whole method of the separation of symbols is founded. It is not necessary therefore to repeat the process of reasoning by which we arrive at the conclusion, that $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$ may be expanded into a sum of partial operations, the same in form as those into which $\{f(z)\}^{-1}$ may be resolved. Still less necessary is it to work out the actual result by employing the symbol $\frac{d}{dx}$ in place

of z; it is quite sufficient to assume the form of the result as known, and to make use of our previous knowledge for the simplification of the problem. Let us therefore at once assume that $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$ may be resolved into a series of simple inverse operations,

$$N_1\left(\frac{d}{dx}-a_1\right)^{-1}+N_2\left(\frac{d}{dx}-a_2\right)^{-1}+N_3\left(\frac{d}{dx}-a_3\right)^{-1}+\&c.$$

where a_1, a_2, a_3, &c. are the roots of the equation

$$f(z)=0.$$

Various methods may be employed to determine the coefficients N_1, N_2, &c., as may be seen in any work on Algebra. We shall not consider these at all, but shall content ourselves with assuming that

$$N_1=\frac{1}{(a_1-a_2)(a_1-a_3)\ldots\ldots(a_1-a_n)},$$

with similar forms for N_2, N_3, &c. These being supposed to be known, the resolution of $\left(f\frac{d}{dx}\right)^{-1}$ is complete, and there merely remains the transformation of the operation $\left(\frac{d}{dx}-a\right)^{-1}$ into the form of an integral by means of the theorem

$$\left(\frac{d}{dx}-a\right)^n=\varepsilon^{ax}\left(\frac{d}{dx}\right)^n\varepsilon^{-ax}.$$

Applying these principles, therefore, to the resolution of the equation (2), we obtain as our result

$$y=N_1\varepsilon^{a_1x}\int\varepsilon^{-a_1x}X\,dx+N_2\varepsilon^{a_2x}\int\varepsilon^{-a_2x}X\,dx+\&c.\ldots\ldots(3),$$

which is the simplest and most symmetrical form into which the solution of the equation can be brought.

If we suppose r of the roots of the equation

$$f(z)=0,$$

to be equal to each other, then the resolution into simple operations will give rise to an expression of the form

$$M\left(\frac{d}{dx}-a\right)^{-r}X+M_1\left(\frac{d}{dx}-a\right)^{-(r-1)}X+\&c.+M_{r-1}\left(\frac{d}{dx}-a\right)^{-1}X$$
$$+N_1\left(\frac{d}{dx}-a_1\right)^{-1}X+N_2\left(\frac{d}{dx}-a_2\right)^{-1}X+\&c.$$

the terms in the second line being those arising from the unequal roots. The coefficients M, M_1, &c., can easily be determined by the usual process of differentiation; the result of which would give us, if we suppose $f(z)=(x-a)^r\phi(z)$,

$$M_p=\frac{1}{1\cdot2\ldots p}\left(\frac{d}{dz}\right)^p\{\phi(z)\}^{-1},\quad\text{when}\quad z=a.$$

The forms of these coefficients then being known, we have

$$y = \varepsilon^{ax}\{M\int^r \varepsilon^{-ax} X\,dx^r + M_1\int^{r-1}\varepsilon^{-ax} X\,dx^{r-1} + \&c.\}$$
$$+ N_1\varepsilon^{a_1x}\int\varepsilon^{-a_1x} X\,dx + N_2\varepsilon^{a_2x}\int\varepsilon^{-a_2x} X\,dx + \&c. \ldots\ldots (4).$$

If the equation $f(z) = 0$ involve impossible roots, as these enter by pairs, the corresponding pairs of binomial operations may be always reduced, so that the result of the operations shall involve possible circular functions instead of impossible exponentials. Let

$$\alpha + \beta\sqrt{-1}, \quad \alpha - \beta\sqrt{-1},$$

be a pair of impossible roots, which for the sake of generality we shall suppose to be repeated r times in $f(z) = 0$; then the solution will consist of a series of pairs of terms, the coefficient of one term in each pair being the same function of $\alpha + \beta\sqrt{-1}$ that the other is of $\alpha - \beta\sqrt{-1}$. By the theory of impossible quantities these coefficients may be represented by $C + D\sqrt{-1}$ and $C - D\sqrt{-1}$, so that the series may be represented by

$$\Sigma\{(C + D\sqrt{-1})\,\varepsilon^{(\alpha+\beta\sqrt{-1})x}\int^p \varepsilon^{-(\alpha+\beta\sqrt{-1})x} X\,dx^p\}$$
$$+ \Sigma\{(C - D\sqrt{-1})\,\varepsilon^{(\alpha-\beta\sqrt{-1})x}\int^p \varepsilon^{-(\alpha-\beta\sqrt{-1})x} X\,dx^p\},$$

where the Σ refers to p, which is to receive all integer values from 1 up to r.

Now
$$\varepsilon^{\beta x\sqrt{-1}} = \cos\beta x + \sqrt{-1}\sin\beta x,$$
$$\varepsilon^{-\beta x\sqrt{-1}} = \cos\beta x - \sqrt{-1}\sin\beta x.$$

Consequently the two sums are reduced to

$$2\Sigma\,.\,\varepsilon^{\alpha x}\begin{cases}(C\cos\beta x - D\sin\beta x)\int^p \varepsilon^{-\alpha x}\cos\beta x\,X\,dx^p \\ +(C\sin\beta x + D\cos\beta x)\int^p \varepsilon^{-\alpha x}\sin\beta x\,X\,dx^p\end{cases} \ldots (5),$$

where the index of integration receives every value from 1 up to r.

As an example of the application of this method, let us take the equation

$$\frac{d^n y}{dx^n} - y = X,$$

n being even.

The factors of $z^n - 1 = 0$ are all included in the formula $\cos\theta \pm \sqrt{-1}\sin\theta$, where

$$\theta = \frac{2\lambda\pi}{n},$$

λ receiving all values from 0 to $\frac{n}{2}$.

Therefore any pair of terms of the solution may be represented by

$$A\left(\frac{d}{dx} - a\right)^{-1} X + B\left(\frac{d}{dx} - b\right)^{-1} X,$$

where $a = \cos\theta + \sqrt{-1}\sin\theta$, $b = \cos\theta - \sqrt{-1}\sin\theta$.

We easily find $A = \frac{a}{n}$, $B = \frac{b}{n}$, so that the expression (5) becomes in this case, since $p = 1$, only

$$\frac{2}{n}[\cos\theta\cos(x\sin\theta) - \sin\theta\sin(x\sin\theta)]\int\varepsilon^{-x\cos\theta}\cos(x\sin\theta)\mathrm{X}dx$$

$$+\frac{2}{n}[\cos\theta\sin(x\sin\theta) + \sin\theta\cos(x\sin\theta)]\int\varepsilon^{-x\cos\theta}\cos(x\sin\theta)\mathrm{X}dx$$

$$= \frac{2}{n}[\cos(x\sin\theta+\theta)\int\varepsilon^{-x\cos\theta}\cos(x\sin\theta)\,\mathrm{X}dx$$

$$+ \sin(x\sin\theta+\theta)\int\varepsilon^{-x\cos\theta}\sin(x\sin\theta)\,\mathrm{X}dx].$$

Precisely the same mode of procedure is applicable to equations in Finite Differences. If we have an equation of the form

$$u_{x+n} + \mathrm{A}_1 u_{x+n-1} + \mathrm{A}_2 u_{x+n-2} + \ldots\ldots + \mathrm{A}_n u_x = \mathrm{X},$$

and introduce (as Mr. Gregory has done, Vol. I. p. 55.) a symbol D characterized by the property

$$\mathrm{D}u_x = u_{x+1},\ \mathrm{D}u_{x+1} = u_{x+2},\ \&\mathrm{c.},$$

our equation assumes the form

$$(\mathrm{D}^n + \mathrm{A}_1\mathrm{D}^{n-1} + \mathrm{A}_2\mathrm{D}^{n-2} + \ldots\ldots + \mathrm{A}_n)\,u_x = \mathrm{X}.$$

Whence we find

$$u_x = (\mathrm{D}^n + \mathrm{A}_1\mathrm{D}^{n-1} + \mathrm{A}_2\mathrm{D}^{n-2} + \ldots\ldots + \mathrm{A}_n)^{-1}\,\mathrm{X}.$$

Then by the method of rational fractions this can be decomposed into a sum of binomial operations of the form

$$u_x = \mathrm{N}_1(\mathrm{D}-a_1)^{-1}\mathrm{X} + \mathrm{N}_2(\mathrm{D}-a_2)^{-1}\mathrm{X} + \mathrm{N}_3(\mathrm{D}-a_3)^{-1}\mathrm{X} + \ldots\ldots$$

where N_1, N_2, &c. have the same form as the similar coefficients in the solution of the differential equation, and a_1, a_2, a_3, &c. are the roots of

$$z^n + \mathrm{A}_1 z^{n-1} + \ldots\ldots + \mathrm{A}_n = 0.$$

Now by the theorem, Vol. i. p. 55, we have

$$(\mathrm{D}-a)^n\,\mathrm{X} = a^{x+n}\,\Delta^n(\mathrm{X}a^{-x}),$$

so that the solution becomes

$$u_x = \mathrm{N}_1 a_1^{x-1}\,\Sigma(\mathrm{X}a_1^{-x}) + \mathrm{N}_2 a_2^{x-1}\,\Sigma(\mathrm{X}a_2^{x-1}) + \ldots\ldots(6.)$$

If there be r roots equal to a, we shall obtain by the same method as that employed in the corresponding case of differential equations

$$u_x = \mathrm{M}a^{x-n}\,\Sigma^n(\mathrm{X}a^{-x}) + \mathrm{M}_1 a^{x-n+1}\,\Sigma^{n-1}(\mathrm{X}a^{-x}) +$$
$$+ \text{ terms involving the unequal roots}\ldots\ldots(7).$$

If there be r pairs of impossible roots, we shall find, by adapting to the preceding expression the notation and reasoning previously employed, the following expression for the general term of u_x so far as it depends on the equal impossible roots

$$(\mathrm{C}+\mathrm{D}\sqrt{-1})(\alpha+\beta\sqrt{-1})^{x-p}\,\Sigma^p\{\mathrm{X}(\alpha+\beta\sqrt{-1})^{-x}\}$$
$$+ (\mathrm{C}-\mathrm{D}\sqrt{-1})(\alpha-\beta\sqrt{-1})^{x-p}\,\Sigma^p\{\mathrm{X}(\alpha-\beta\sqrt{-1})^{-x}\}.$$

To reduce this into a more convenient shape, let us assume

$$\alpha = \rho \cos\theta, \quad \beta = \rho \sin\theta,$$

whence we have

$$(\alpha+\beta\sqrt{-1})^{x-p} = \rho^{x-p}\{\cos(x-p)\theta+\sqrt{-1}\sin(x-p)\theta\},$$

and similarly for the other expressions.

Substituting these values and reducing, we obtain for the general term of u_x

$$2\rho^{x-p}\left\{\begin{array}{l} \mathrm{C}\cos(x-p)\theta-\mathrm{D}\sin(x-p)\theta.\Sigma^p[\rho^{-x}\mathrm{X}\cos(x\theta)] \\ +\mathrm{C}\sin(x-p)\theta+\mathrm{D}\cos(x-p)\theta.\Sigma^p[\rho^{-x}\mathrm{X}\sin(x\theta)] \end{array}\right. \quad \ldots(8.)$$

The complete integral, so far as it depends on the pairs of impossible roots, will consist of a series of terms similar to the above, in which the index p receives every value from 1 to r.

The analogy between (3), (4), (5), and (6), (7), (8), is very remarkable, and unless we employed a method of solution common to both problems, it would not be easy to see the reason for so close a resemblance in the solution of two different kinds of equations. But the process which I have here exhibited shows, that the form of the solution depends solely on the method of decomposing the original operating factor; and this decomposition is effected by means of processes which are common to the two operations under consideration, being founded only on the common laws of the combinations of the symbols.

It is thus seen that every step of the solution of Differential Equations and equations of Finite Differences is reduced to the known theorems of ordinary Algebra, with the exception of the two theorems

$$\left(\frac{d}{dx}-a\right)^n \mathrm{X} = \varepsilon^{ax}\left(\frac{d}{dx}\right)^n \varepsilon^{-ax}\mathrm{X}, \text{ and } (\mathrm{D}-a)^n\mathrm{X} = a^{x+n}\Sigma^n(a^{-x}\mathrm{X}),$$

which are necessary for passing to the interpretation of the expressions at which we arrive. This seems to be as great a simplification of the problem as the present state of mathematics admits of, for any further improvement must involve the invention of new processes for the treatment of ordinary algebraical expressions. With such we are not at present concerned; our object being to reduce the more complicated processes of the higher analysis to the simpler results which have been already obtained, and which may be looked on in the military phrase as bases for our further operations.

Waddington, near Lincoln.

VI.—INVESTIGATION OF THE ABERRATION IN RIGHT ASCENSION AND DECLINATION.

THE following investigation of the formulæ for Aberration in Right Ascension and Declination, will be found to be more simple than that given in Maddy's *Astronomy*, p. 214.

Let γL (fig. 4.) be the ecliptic, γE the equator, P its pole, T a point 90^0 behind the place of the Sun, S the place of the star; then ST will be the plane of aberration.

Let $\gamma N = a$, $PN = \delta$, Sun's longitude $= \odot$, and $L\gamma E = \omega$.

1st. For the Aberration in Declination: produce SN to a point Q, such that $SQ = 90^0$, and join TQ, TN. If A be the coefficient of aberration, and $\Delta\delta$ the aberration in declination,

$$\Delta\delta = -A \sin ST \cdot \cos TSQ = -A \cos TQ,$$

as TSQ is a quadrantal triangle.

But $\cos QT = \cos TN \cdot \cos QN + \sin TN \cdot \sin QN \cdot \sin TN\gamma$,

and $\cos TN = \cos T\gamma \cdot \cos \gamma N + \sin T\gamma \cdot \sin \gamma N \cdot \cos T\gamma N$,

$$= \sin \odot \cos a - \cos \odot \sin a \cos \omega.$$

Also, $\sin TN \cdot \sin TN\gamma = \sin T\gamma \cdot \sin T\gamma N = \cos \odot \sin \omega$.

Substituting these values, and putting $90^0 - \delta$ for QN, we find

$$\Delta\delta = -A \{\sin \delta (\sin \odot \cos a - \cos \odot \sin a \cos \omega) + \cos \delta \cos \odot \sin \omega\}.$$

2nd. For the Aberration in Right Ascension: produce $N\gamma$ to a point R, such that $NR = 90^0$, and join RS, RT. Then, if Δa be the aberration in right ascension,

$$\Delta a = -\frac{A}{\cos \delta} \sin ST \cdot \cos TSR = -\frac{A}{\cos \delta} \cos RT.$$

But $\cos RT = \cos T\gamma \cdot \cos R\gamma + \sin T\gamma \cdot \sin R\gamma \cos R\gamma T$,

$$= \sin \odot \sin a + \cos a \sin \odot \cos \omega.$$

Consequently,

$$\Delta a = -\frac{A}{\cos \delta} \{\sin \odot \sin a + \cos a \sin \odot \cos \omega\}.$$

R. L. E.

VII.—ON THE SYMPATHY OF PENDULUMS.

BY the Sympathy of Pendulums is meant the effect on the motions of different pendulums produced by their mutual action, when their points of suspension have any elastic or moveable connexion.

The phenomenon is a striking one, and presents itself in a marked manner to those who are engaged in the art of clock-making. It has been observed by them, that if the pendulums of two clocks, the times of the oscillation of which are different, be so situated that the motion of the one can be in any way communicated to the other—as, for instance, by their centres of suspension being attached to the same beam—the motions of the two pendulums are entirely altered by their mutual action, the periods of both tending to become the same, and the extent of oscillation continually changing. This becomes a serious practical inconvenience, and it is necessary to take precautions to prevent the influence of the one pendulum being communicated to the other. Daniel Bernoulli appears to have been the first who took notice of this phenomenon, at least with any reference to theory; but the case which attracted his attention was far more simple than that to which we have alluded. It was that of the motion of the two scales of a balance, when one has had an oscillatory motion communicated to it. The following is his narration of the phenomenon as he observed it. (*Nova Commen. Petrop.*, Vol. xix. p. 281.)

"Cum aliquando in libra, majori eaque subpigra, alteram lancem "forte fortuna ad latus diducerem, moxque rursus dimitterem, "accidit utique ut protinus hinc inde oscillaret nec ab initio lanx "opposita de loco moveretur: mox autem et hæc quoque agitari "sensimque majores oscillationes formare, dum e contrario lanx "prior motum suum oscillatorium gradatim perderet tandemque "fere quiesceret; hoc ipso momento altera maximum motionis "gradum, initiali lancis sociæ fere æqualem, attingebat: tunc "ordine contrario eædem mutationes repetebantur, usque dum "prima lanx motum suum primitivum integrum resumeret so- "ciaque quieti ad momentum redderetur; hæc autem oscillationum "communicatio ac reciprocatio diu satis sese manifestabat."

Bernoulli does not seem to have attempted a direct solution of the dynamical problem which this experiment suggested, but contents himself with adducing it as an instance in support of his principle of the coexistence of small oscillations. In the same volume of the *Petersburgh Memoirs,* however, Euler in two papers considers the question theoretically. It is clear that the experiment of Bernoulli admits of being performed in two different ways: the original displacement of the scale may either be in the vertical plane passing through the beam of the balance, or it may be in any other plane. Euler only considers the first case, though another one—that when the original displacement is perpendicular to the vertical plane passing through the beam—is also interesting, and admits of as easy a solution. In his first memoir, Euler supposes that the centre of suspension of the balance is in the same line as the centres of suspension of the two scales; and on investigating the result on this supposition, he finds that the interchange of motion described by Bernoulli could not take place, though the motion of the scale originally put in motion would be different from that

which it would have if suspended from a fixed point of support. This result is confirmed by the simple consideration, that when the centre of suspension of the beam is in the line joining the centres of suspension of the scales, these will only receive vertical motions from the motion of the beam; and consequently, if we suppose the second scale to be originally at rest, it will have no horizontal motion communicated to it, so as to cause it to oscillate in a horizontal direction.

In the other memoir Euler considers the case of the centre of suspension being above the line joining the points of suspension, as indeed would be the case in an ordinary balance which is usually suspended by some higher point. This investigation we shall here give, adhering pretty closely to the process adopted by Euler, as it is always both interesting and instructive to see the mode in which the first writers attacked such a problem as this.

Let O (fig. 5.) be the point of suspension of the whole balance, G its centre of gravity, AB the beam, P and Q the scales, which are here supposed to be material points. Draw aOb horizontal, Aα, Bβ vertical. Let AC=CB=a, OC=b, OG=c, AP=BQ=l, Mk^2 = moment of inertia of the beam, m the mass of P and of Q supposed to be equal.

Let φ be the angle which, at the time t, the beam makes with the horizon.

$$\text{Let} \quad \mathrm{PA}\alpha = \eta, \quad \mathrm{QB}\beta = \theta,$$

$$Op = x, \quad Pp = y, \quad Oq = x', \quad Qq = y'.$$

Then

$$x = a\cos\phi + b\sin\phi + l\sin\eta,$$
$$y = b\cos\phi - a\sin\phi + l\cos\eta,$$
$$x' = a\cos\phi - b\sin\phi - l\sin\theta,$$
$$y' = a\sin\phi + b\cos\phi + l\cos\theta.$$

Let P, Q, be the tensions of AP and BQ.

The equations of their motions are

$$(1)\ \frac{d^2x}{dt^2} = -\frac{Pg\sin\eta}{m}. \qquad (2)\ \frac{d^2y}{dt^2} = g - \frac{Pg\cos\eta}{m}.$$

$$(3)\ \frac{d^2x'}{dt^2} = \frac{Qg\sin\theta}{m}. \qquad (4)\ \frac{d^2y'}{dt^2} = g - \frac{Qg\cos\theta}{m}.$$

For the motion of the beam we have its own weight at G tending to turn it back to its original position, and the tensions of the strings acting in different directions. The moment of the couple arising from the weight of the beam is

$$\mathrm{M}gc\cos\mathrm{GOA} = \mathrm{M}gc\sin\phi.$$

The moment of the couple arising from the tension P is

$$Pg\,Om = Pg\,\{a\cos(\eta-\phi) - b\sin(\eta-\phi)\}.$$

Both these are negative, as tending to bring back the beam to its original position. The moment of the couple arising from Q is

$$Qg\,\{a\cos(\phi-\theta) - b\sin(\phi-\theta)\}.$$

Consequently we have, for the motion of the beam, the equation

$$(5)\quad \frac{d^2\phi}{dt^2} = -\frac{g}{Mk^2}\left\{\begin{array}{l} Mc\sin\phi + P\,[a\cos(\eta-\phi) - b\sin(\eta-\phi)] \\ \qquad -Q\,[a\cos(\phi-\theta) - b\sin(\phi-\theta)] \end{array}\right\}$$

These five simultaneous equations, if solved, would serve to determine all the circumstances of the motion, but under their present form the solution is impracticable. To render it possible, we must suppose the displacements to be very small, so that we may put the arc for its sine and unity for the cosine; by this means we find

$$x = a + b\phi + l\eta, \qquad y = b - a\phi + l,$$
$$x' = a - b\phi - l\theta, \qquad y' = a\phi + b + l.$$

Also, the tensions of the strings may be supposed not to be changed, but to remain equal to the weight. Making these substitutions in the five equations of motion, the second and fourth disappear, and there remain

$$b\frac{d^2\phi}{dt^2} + l\frac{d^2\eta}{dt^2} = -g\eta,$$

$$b\frac{d^2\phi}{dt^2} + l\frac{d^2\theta}{dt^2} = -g\theta,$$

$$Mk^2\frac{d^2\phi}{dt^2} = -g\{(Mc + 2mb)\phi - mb(\eta+\theta)\},$$

and by means of these three simultaneous equations we can easily determine ϕ, η, θ.

Let $\frac{g}{l} = n^2$, $\frac{b}{l} = h$, $\frac{Mc + 2mb}{Mk^2}g = p^2$, $-\frac{mbg}{Mk^2} = q$.

Then the equations may be put under the form

$$(1)\quad h\frac{d^2\phi}{dt^2} + \left(\frac{d^2}{dt^2} + n^2\right)\eta = 0.$$

$$(2)\quad h\frac{d^2\phi}{dt^2} + \left(\frac{d^2}{dt^2} + n^2\right)\theta = 0.$$

$$(3)\quad \left(\frac{d^2}{dt^2} + p^2\right)\phi + q(\eta+\theta) = 0.$$

*Add together (1) and (2), which gives

$$(4)\quad 2h\frac{d^2\phi}{dt^2} + \left(\frac{d^2}{dt^2} + n^2\right)(\eta+\theta) = 0.$$

Subtract (2) from (1), which gives

$$(5)\quad \left(\frac{d^2}{dt^2} + n^2\right)(\eta - \theta) = 0.$$

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* For this method of integrating simultaneous differential equations, see *Mathematical Journal*, Vol. I., p. 173.

Operate on (3) with $\left(\frac{d^2}{dt^2} + n^2\right)$, multiply (4) by q, and subtract it from the former. Then

$$\left(\frac{d^2}{dt^2} + n^2\right)\left(\frac{d^2}{dt^2} + p^2\right)\phi - 2hq\,\frac{d^2\phi}{dt^2} = 0.$$

If $-\mu_1^2$, $-\mu_2^2$, be the roots of

$$z^2 + (n^2 + p^2 - 2hq)\,z + n^2p^2 = 0,$$

this may be put under the form

$$(6)\quad \left(\frac{d^2}{dt^2} + \mu_1^2\right)\left(\frac{d^2}{dt^2} + \mu_2^2\right)\phi = 0.$$

The integral of which is

$$(A)\quad \phi = C_1 \cos(\mu_1 t + a_1) + C_2 \cos(\mu_2 t + a_2).$$

Substituting this value of ϕ in (3), we find

$$\eta + \theta = \frac{C_1}{q(\mu_1^2 - p^2)} \cos(\mu_1 t + a_1) + \frac{C_2}{q(\mu_2^2 - p^2)} \cos(\mu_2 t + a_2).$$

Integrating (5), we have

$$\eta - \theta = C \cos(nt + a);$$

whence

$$(B)\quad 2\eta = \frac{C_1}{q(\mu_1^2 - p^2)} \cos(\mu_1 t + a_1) + \frac{C_2}{q(\mu_2^2 - p^2)} \cos(\mu_2 t + a_2) + C \cos(nt + a),$$

$$(C)\quad 2\theta = \frac{C_1}{q(\mu_1^2 - p^2)} \cos(\mu_1 t + a_1) + \frac{C_2}{q(\mu_2^2 - p^2)} \cos(\mu_2 t + a_2) - C \cos(nt + a).$$

The equations (A), (B), (C), are the complete solution of the problem, and involve six arbitrary constants, viz. C, C_1, C_2, a, a_1, a_2, which may be determined so as to suit the original circumstances, and according to the nature of these the solution will assume different shapes. To suit the experiment of Bernoulli, we must suppose, when $t = 0$, that

$$\phi = 0, \qquad \eta = \varepsilon, \qquad \theta = 0,$$

$$\frac{d\phi}{dt} = 0, \qquad \frac{d\eta}{dt} = 0, \qquad \frac{d\theta}{dt} = 0.$$

The last three conditions give us $a = a_1 = a_2 = 0$; and these values substituted in the others reduce them to

$$0 = C_1 + C_2, \qquad \varepsilon = C,$$

$$q\varepsilon = \frac{C_1}{\mu_1^2 - p^2} + \frac{C_2}{\mu_2^2 - p^2};$$

whence we find

$$C_1 = q\varepsilon\,\frac{(\mu_1^2 - p^2)(\mu_2^2 - p^2)}{\mu_2^2 - \mu_1^2}, \qquad C_2 = q\varepsilon\,\frac{(\mu_1^2 - p^2)(\mu_2^2 - p^2)}{\mu_1^2 - \mu_2^2}.$$

Substituting these values the equations become

$$\phi = q\,\frac{\varepsilon(\mu_1^2-p^2)(\mu_2^2-p^2)}{\mu_2^2-\mu_1^2}(\cos\mu_1 t-\cos\mu_2 t),$$

$$2\eta = \frac{\varepsilon}{\mu_2^2-\mu_1^2}\{(\mu_2^2-p^2)\cos\mu_1 t-(\mu_1^2-p^2)\cos\mu_2 t\}+\varepsilon\cos nt,$$

$$2\theta = \frac{\varepsilon}{\mu_2^2-\mu_1^2}\{(\mu_2^2-p^2)\cos\mu_1 t-(\mu_1^2-p^2)\cos\mu_2 t\}-\varepsilon\cos nt.$$

Observing that $\mu_1^2+\mu_2^2 = n^2+p^2-2hq$, and $\mu_1^2\mu_2^2 = n^2p^2$, the values of C_1 and C_2 are reduced to

$$C = \frac{2\varepsilon hp^2q^2}{\mu_2^2-\mu_1^2}, \qquad C_2 = \frac{2\varepsilon hp^2q^2}{\mu_1^2-\mu_2^2};$$

so that we find

$$\phi = \frac{2\varepsilon hp^2q^2}{\mu_2^2-\mu_1^2}(\cos\mu_1 t-\cos\mu_2 t),$$

$$2\eta = \frac{2\varepsilon hp^2q}{\mu_2^2-\mu_1^2}\left(\frac{\cos\mu_1 t}{\mu_1^2-p^2}-\frac{\cos\mu_2 t}{\mu_2^2-p^2}\right)+\varepsilon\cos nt,$$

$$2\theta = \frac{2\varepsilon hp^2q}{\mu_2^2-\mu_1^2}\left(\frac{\cos\mu_1 t}{\mu_1^2-p^2}-\frac{\cos\mu_2 t}{\mu_2^2-p^2}\right)-\varepsilon\cos nt.$$

It appears from these expressions, that the motion of the beam is compounded of two oscillations of different periods, $\frac{2\pi}{\mu_1}$ and $\frac{2\pi}{\mu_2}$. The relations which these oscillations bear to those which the beam and the scales would separately make, may be easily shown. By means of equations (3) and (4), we see that μ_1^2 and μ_2^2 must satisfy the equation

$$(\mu^2-n^2)(\mu^2-p^2)+2hq\mu^2 = 0.$$

If the mass of the beam be very large, we may suppose p^2 to be less than n^2, and the equation for μ^2 may be put under the forms

$$\mu_1^2 = n^2-\frac{2hq\mu_1^2}{\mu_1^2-p^2},$$

$$\text{and}\quad \mu_2^2 = p^2+\frac{2hq\mu_2^2}{n^2-\mu_2^2},$$

which show that μ_1^2 is less than n^2, and μ_2^2 greater than p^2.

Hence it appears that the period of the one part of the oscillation of the beam is greater than the period of the natural oscillation of the scales, while the period of the other oscillation is less than that of the beam and scales considered as one mass. The motion of the scales consists of these same oscillations, with the addition of one, the period of which is that of a pendulum of the same length as the suspending string.

If we suppose the vibrations of the scales to take place in a plane perpendicular to a vertical plane passing through the beam,

the expressions become somewhat simpler. We shall not, in this case, go so much into detail as in the last, but shall at once suppose the displacements to be so small, that the forces of restitution may be considered as proportional to them.

Let AB (fig. 6.) be the original position of the beam, PQ its position at the time t; p, q the projections of the positions of scales considered as material points at the same time.

Let $AC = BC = a$, $AP = BQ = z$, $Pp = x$, $Qq = y$, Mk^2 be the moment of inertia of the beam round C, m the mass of each weight, l the length of the string by which each weight is suspended. Then the equations of motion will be

$$\frac{d^2(x+z)}{dt^2} + \frac{g}{l}x = 0 \quad \ldots\ldots\ldots\ldots (1),$$

$$\frac{d^2(y+z)}{dt^2} + \frac{g}{l}y = 0 \quad \ldots\ldots\ldots\ldots (2),$$

$$\frac{d^2z}{dt^2} - \frac{g}{l}\frac{ma^2}{Mk^2}(x+y) = 0 \quad \ldots\ldots\ldots (3).$$

Subtracting (2) from (1), we have

$$\frac{d^2(x-y)}{dt^2} + \frac{g}{l}(x-y) = 0.$$

Add (1) and (2), and subtract (3) multiplied by 2; then

$$\frac{d^2(x+y)}{dt^2} + \frac{g}{l}\left(1 + 2\frac{ma^2}{Mk^2}\right)(x+y) = 0.$$

Let $\frac{g}{l} = n^2$, $\frac{g}{l}\left(1+2\frac{ma^2}{Mk^2}\right) = n'^2$; then these equations become

$$\frac{d^2(x-y)}{dt^2} + n^2(x-y) = 0,$$

$$\frac{d^2(x+y)}{dt^2} + n'^2(x+y) = 0.$$

Integrating, we have

$$x - y = C\cos(nt + \alpha),$$
$$x + y = C_1\cos(n't + \alpha_1).$$

If we suppose that at the beginning of the motion

$$x = c, \quad y = 0, \quad \frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0,$$

these equations become

$$x - y = C\cos nt,$$
$$x + y = C\cos n't;$$

whence
$$x = C\cos\frac{n'-n}{2}t\cos\frac{n'+n}{2}t,$$
$$y = -C\sin\frac{n'-n}{2}t\sin\frac{n'+n}{2}t.$$

Substituting the value of $x + y$ which has been found in (3), and integrating, we find

$$z = -\frac{g}{l}\,\frac{ma^2}{Mk^2}\,\frac{c}{n'^2}\cos n't + At + B.$$

If at the beginning of the motion we suppose $z = 0$, $\frac{dz}{dt} = 0$, we shall find $A = 0$, $B = \frac{g}{l}\,\frac{ma^2}{Mk^2}\,\frac{c}{n'^2}$, so that

$$z = \frac{g}{l}\,\frac{ma^2}{Mk^2}\,\frac{c}{n'^2}(1 - \cos n't),$$

$$\text{or}\quad z = \frac{2g}{l}\,\frac{ma^2}{Mk^2}\,\frac{c}{n'^2}\sin^2\frac{n't}{2}.$$

If we suppose the beam to have an original angular velocity given to it, then A will not be 0, and the expression for z will no longer be simply periodic, but will increase continually with the time. This will also appear from the consideration, that there is no force independent of the oscillations of the scale acting on the beam, so that any originally impressed velocity will not be destroyed, but will continue to carry the beam round with a motion subject to periodic inequalities.

If we suppose the mass of the beam to be very great in comparison with the masses of the weights, so that $\frac{ma^2}{Mk^2}$ is very small, n is very nearly equal to n', $\frac{n' - n}{2}$ is very small, and $\sin\frac{n' - n}{2}t$ and $\cos\frac{n' - n}{2}t$ vary very slowly.

So that we may represent our result as that of two pendulums, whose arcs of vibration are respectively

$$C\cos\frac{n' - n}{2}t \text{ and } C\sin\frac{n' - n}{2}t.$$

These are complementary; they show that the arc of vibration of the first pendulum will gradually diminish, and that of the second increase, till after a time $= \frac{2}{n' - n}\cdot\frac{\pi}{2}$ they have interchanged motions and the converse process is repeated, and the system returns to its original state after a time $= \frac{2\pi}{n' - n}$.

The common time of oscillation is that of a pendulum whose length is

$$g\cdot\frac{4}{(n + n')^2} = l\left(1 - \frac{ma^2}{Mk^2}\right) \text{ nearly.}$$

The most general case of the influence of one pendulum on another, when the motions as we have supposed are all in the same horizontal direction and infinitesimal, will be when, calling A and B the points of support, each of these when disturbed performs vibrations in known times; and a disturbance given to A communicates a known motion to B, and *vice versâ*.

To investigate the motion in this case, let u, v be the coordinates of A and B, $u+x$, $v+y$ of the balls suspended to them; then we have the equations

$$\frac{d^2(u+x)}{dt^2} + m^2x = 0,$$

$$\frac{d^2(v+y)}{dt^2} + n^2y = 0,$$

$$\frac{d^2u}{dt^2} + p^2u - ax - fv = 0,$$

$$\frac{d^2v}{dt^2} + q^2v - by - gu = 0.$$

A solution of these equations is

$$x = \mathrm{RM}\cos(\sqrt{\rho}\,t - r), \qquad u = \mathrm{RP}\cos(\sqrt{\rho}\,t - r),$$

$$y = \mathrm{RN}\cos(\sqrt{\rho}\,t - r), \qquad v = \mathrm{RQ}\cos(\sqrt{\rho}\,t - r);$$

and to determine ρ we get the equation

$$\{(\rho - m^2)(\rho - p') - a\rho\}\{(\rho - n^2)(\rho - q^2) - b\rho\} - fg(\rho - m^2)(\rho - n^2) = 0.$$

The four values of ρ determined from this equation are all positive, and therefore the angular functions real. The complete solution is the sum of the particular solutions, therefore

$$x = \mathrm{R_1M_1}\cos(\sqrt{\rho_1}\,t - r_1) + \mathrm{R_2M_2}\cos(\sqrt{\rho_2}\,t - r_2) + \mathrm{R_3M_3}\cos(\sqrt{\rho_3}\,t - r_3) + \mathrm{R_4M_4}\cos(\sqrt{\rho_4}\,t - r_4),$$

$$y = \mathrm{R_1N_1}\cos(\sqrt{\rho_1}\,t - r_1) + \mathrm{R_2N_2}\cos(\sqrt{\rho_2}\,t - r_2) + \mathrm{R_3N_3}\cos(\sqrt{\rho_3}\,t - r_3) + \mathrm{R_4N_4}\cos(\sqrt{\rho_4}\,t - r_4).$$

If three of the quantities R are equal to zero, the vibrations of the two pendulums are isochronous, and there are therefore four modes of this isochronous vibration. In all cases the pendulums affect each other, so that none of the points oscillates in its natural time. The extreme generality of the equations we have assumed, and consequently of the solution derived from them, prevents us from interpreting our result in a more precise manner. For this purpose it would be necessary to assign some relations between the constants in the equations, but it would lead us too far if we were to attempt any such investigation; and we may add, that any particular case will in general be more easily solved by a direct reference to its own circumstances than by a reduction of the general solution.

D. G. S.

VIII.—ON THE EXPANSION OF COSINES AND SINES OF MULTIPLE ARCS IN ASCENDING POWERS OF THE COSINES AND SINES OF THE SIMPLE ARCS.

THE method adopted by Lagrange for expressing the cosine and sine of multiple arcs in terms of the powers of the cosines and sines of the simple arcs, depended on the expansion of functions of the form

$$(x+\sqrt{x^2-1})^n \text{ and } (x-\sqrt{x^2-1})^n.$$

The same method is pursued by Poinsot in his *Recherches sur l'Analyse des Sections Angulaires*, where the complete theory of these circular functions was first given; but he has also indicated another way, which is far less tedious and complicated—that of assuming the form of the series, and determining the coefficients by differentiation. As this may be useful to those who are studying the subject, we shall here briefly fill up the outline which Poinsot has sketched

1. To expand $\cos n\theta$ in terms of $\cos\theta$ and its powers.

Assume

$$\cos n\theta = a_0 + a_1\cos\theta + \ldots + a_p(\cos\theta)^p + \ldots + a_{p+2}(\cos\theta)^{p+2} + \ldots$$

Differentiating,

$$n\sin n\theta = \{a_1 + 2a_2\cos\theta + \ldots + pa_p(\cos\theta)^{p-1} + \ldots + (p+2)\,a_{p+2}(\cos\theta)^{p+1} + \ldots\}\sin\theta.$$

Differentiating again,

$$n^2\cos n\theta = a_1\cos\theta + \ldots\ldots + pa_p(\cos\theta)^p + \ldots\ldots + (p+2)\,a_{p+2}(\cos\theta)^{p+2} + \ldots\ldots$$
$$-\{2a_2 + \ldots\ldots + p(p-1)\,a_p(\cos\theta)^{p-2} + \ldots\ldots + (p+1)(p+2)\,a_{p+2}(\cos\theta)^p \ldots\ldots\}\sin^2\theta.$$

Putting $1-\cos^2\theta$ for $\sin^2\theta$, and taking the coefficient of $(\cos\theta)^p$, we find it to be

$$pa_p + p(p-1)\,a_p - (p+1)(p+2)\,a_{p+2},$$

and this must be equal to the coefficient of $(\cos\theta)^p$ in the first equation multiplied by n^2. Therefore, we have

$$n^2 a_p = pa_p + p(p-1)\,a_p - (p+1)(p+2)\,a_{p+2};$$

$$\text{whence } a_{p+2} = -\frac{(n^2-p^2)}{(p+1)(p+2)}\,a_p.$$

By this means any coefficient is found in terms of that two places below it. Consequently the first and second coefficients are left to be determined by other means. For this purpose let $\theta = (2r+1)\frac{\pi}{2}$

in the first equation, r being any integer. Every term on the second side vanishes except the first, and we find

$$a_0 = \cos n\,(2r+1)\,\frac{\pi}{2}.$$

To find a_1, make $\theta = (2r+1)\,\frac{\pi}{2}$ in the second equation, when we obtain

$$a_1 = n\,\frac{\sin n\,(2r+1)\,\frac{\pi}{2}}{\sin\,(2r+1)\,\frac{\pi}{2}} = n\cos\,(n-1)\,(2r+1)\,\frac{\pi}{2}.$$

Starting from these values, and giving p successively all the integer values from 0 upwards, and separating the terms involving odd powers of $\cos\theta$ from those involving even powers, we find

$$\cos n\theta = \cos n\,(2r+1)\,\frac{\pi}{2}\left(1-\frac{n^2}{1.2}\,(\cos\theta)^2+\frac{n^2\,(n^2-2^2)}{1.2.3.4}\,(\cos\theta)^4-\&c.\right)$$
$$+n\cos\,(n-1)\,(2r+1)\,\frac{\pi}{2}\left(\cos\theta-\frac{n^2-1^2}{1.2.3}\,(\cos\theta)^3+\&c.\right)$$

When n is an even integer, the second line being multiplied by the cosine of an odd multiple of $\frac{\pi}{2}$ vanishes, and the first line only remains; when n is an odd integer, the first line vanishes, and the second line only remains. When n is a fraction, both lines must be retained, except for particular values of r, which cause the factor of one or other series to vanish.

2. If we assume

$$\sin n\theta = a_0 + a_1\sin\theta + a_2\,(\sin\theta)^2 + \&c. + a_p\,(\sin\theta)^p + \&c.$$

we shall obtain, by the same means as in the previous case, the same equation for determining a_{p+2}, viz.

$$a_{p+2} = -\,\frac{(n^2-p^2)}{(p+1)\,(p+2)}\,a_p.$$

To determine the first two coefficients, make $\theta = r\pi$ in the above equation and its differential. We thus obtain

$$a_0 = \sin nr\pi,$$
$$a_1 = n\,\frac{\cos nr\pi}{\cos r\pi} = n\cos\,(n-1)\,r\pi\,;$$

so that, dividing the series into two parts containing the odd and the even powers, we have

$$\sin n\theta = \sin nr\pi\left(1-\frac{n^2}{1.2}\,(\sin\theta)^2+\frac{n^2\,(n^2-2^2)}{1.2.3.4}\,(\sin\theta)^4-\&c.\right)$$
$$+n\cos\,(n-1)\,r\pi\left(\sin\theta-\frac{n^2-1^2}{1.2.3}\,(\sin\theta)^3+\&c.\right).$$

When n is an integer the first series always vanishes, and the second is positive or negative according as $(n-1)r$ is even or odd. When n is odd the second series terminates; when n is even it goes on to infinity. When n is a fraction, both series are to be retained.

3. If we assume

$$\cos n\theta = a_0 + a_1 \sin\theta + a_2 (\sin\theta)^2 + \&c. + a_p (\sin\theta)^p - \&c.$$

we obtain as before for determining a_{p+2}, the equation

$$a_{p+2} = -\frac{(n^2 - p^2)}{(p+1)(p+2)} a_p.$$

To determine the first two coefficients, make $\theta = r\pi$ in the equation and its differential. Then

$$a_0 = \cos nr\pi,$$

$$a_1 = -\frac{n \sin nr\pi}{\cos r\pi} = -n \sin (n-1) r\pi;$$

so that

$$\cos n\theta = \cos nr\pi \left(1 - \frac{n^2}{1.2} (\sin\theta)^2 + \frac{n^2 (n^2-2^2)}{1.2.3.4} (\sin\theta)^4 \quad \&c.\right)$$

$$-n \sin (n-1) r\pi \left(\sin\theta - \frac{n^2 - 1^2}{1.2.3} (\sin\theta)^3 + \&c.\right)$$

When n is an integer the second series always disappears, and the first series terminates when n is even, and does not terminate when n is odd. When n is a fraction, both series are retained.

4. If we assume

$$\sin n\theta = a_0 + a_1 \cos\theta + \&c. + a_p (\cos\theta)^p + \&c.$$

we find as before as the condition for determining the coefficients,

$$a_{p+2} = -\frac{(n^2 - p^2)}{(p+1)(p+2)} a_p.$$

Make $\theta = (2r+1)\frac{\pi}{2}$ in the equation and its differential. Then

$$a_0 = \sin n (2r+1) \frac{\pi}{2}.$$

$$a_1 = -n \frac{\cos n (2r+1) \frac{\pi}{2}}{\sin (2r+1) \frac{\pi}{2}} = n \sin (n-1)(2r+1)\frac{\pi}{2};$$

whence we find

$$\sin n\theta =$$

$$\sin n(2r+1)\frac{\pi}{2} \left(1 - \frac{n^2}{1.2} (\cos\theta)^2 + n^2 \frac{(n^2-2^2)}{1.2.3.4} (\cos\theta)^2 + \&c.\right)$$

$$+ n \sin (n-1)(2r+1)\frac{\pi}{2} \left(\cos\theta - \frac{n^2 - 1^2}{1.2.3} (\cos\theta)^3 + \&c.\right).$$

When n is an odd integer the first line, when n is even the second line only remains; but when n is fractional both series must be retained. In no case do the series ever terminate.

These four results may be put under a convenient mnemonic form if we make use of a particular notation, by which the laws of these series are assimilated to those of sines and cosines.

Let P_n be such an operation performed on n that

$$P_n^0 = 1, \qquad P_n^1 = n,$$

$$P_n^2 = (n-0)(n+0) = n^2, \qquad P_n^3 = n(n-1)(n+1) = n(n^2-1^2),$$

$$P_n^4 = n^2(n-2)(n+2) = n^2(n^2-2^2), \qquad P_n^5 = n(n^2-1^2)(n^2-3^2),$$

$$\&c. \qquad \&c.$$

from which the law of formation is evident. Then the series of (1) may be put under the form

$$\cos n\theta = \cos n(2r+1)\frac{\pi}{2}\left(1 - \frac{P_n^2}{1.2}(\cos\theta)^2 + \frac{P_n^4}{1.2.3.4}(\cos\theta)^4 - \&c.\right)$$

$$+ \cos(n-1)(2r+1)\frac{\pi}{2}\left(P_n^1\cos\theta - \frac{P_n^3}{1.2.3}(\cos\theta)^3 + \&c.\right);$$

in which it will be seen that the series in the first line follows the law of the cosine of $(P_n \cos\theta)$, and the series in the second line that of the sine of $(P_n \cos\theta)$, so that the expression may be put under the form

$$\cos n\theta = \cos n(2r+1)\frac{\pi}{2}\cos(P_n\cos\theta) + \cos(n-1)(2r+1)\frac{\pi}{2}\sin(P_n\cos\theta)$$

$$= \cos n(2r+1)\frac{\pi}{2}\cos(P_n\cos\theta) \pm \sin n(2r+1)\frac{\pi}{2}\sin(P_n\cos\theta);$$

whence

$$\text{(I).} \quad \cos n\theta = \cos\left\{n(2r+1)\frac{\pi}{2} \mp (P_n\cos\theta)\right\}.$$

By using the same notation with respect to (2), we have

$$\sin n\theta = \sin nr\pi\left(1 - \frac{P_n^2}{1.2}(\sin\theta)^2 + \frac{P_n^4}{1.2.3.4}(\sin\theta)^4 - \&c.\right)$$

$$+ \cos(n-1)r\pi\left(P_n\sin\theta - \frac{P_n^3}{1.2.3}(\sin\theta)^3 + \&c.\right)$$

$$= \sin nr\pi\cos(P_n\sin\theta) \pm \cos nr\pi\sin(P_n\sin\theta);$$

and therefore

$$\text{(II).} \quad \sin n\theta = \sin\left\{nr\pi \pm (P_n\sin\theta)\right\}.$$

Proceeding in the same way, we shall find

$$\cos n\theta = \cos nr\pi \cos (P_n \sin \theta) \mp \sin nr\pi \sin (P_n \sin \theta),$$

or

$$\text{(III).}\quad \cos n\theta = \cos \{nr\pi \pm (P_n \sin \theta)\}.$$

Similarly

$$\text{(IV).}\quad \sin n\theta = \sin \left\{ n(2r+1)\frac{\pi}{2} \mp (P_n \cos \theta) \right\}.$$

G.

IX.—ON THE LINES OF CURVATURE ON AN ELLIPSOID.

By R. L. Ellis, B.A., Trinity College.

The following investigation of the Lines of Curvature on an Ellipsoid, has the advantages of symmetry and of giving a distinct geometrical conception. The artifice on which it depends may, it is thought, be found useful on other occasions.

The symmetrical equation to the lines of curvature is

$$(b^2-c^2)\,x\,dy\,dz + (c^2-a^2)\,y\,dz\,dx + (a^2-b^2)\,z\,dx\,dy = 0 \ldots (1),$$

(see *Mathematical Journal*, Vol. I., p. 142.), where xyz are connected by the equation to the surface,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots (2).$$

$$\text{Put}\quad \frac{x^2}{a^2} = u,\quad \frac{y^2}{b^2} = v,\quad \frac{z^2}{c^2} = w \ldots\ldots\ldots (A).$$

$$\text{Then}\quad x\,dy\,dz = \tfrac{1}{4}a\sqrt{u}\,.\,b\frac{dv}{\sqrt{v}}c\frac{dw}{\sqrt{w}} = \frac{abc}{4\sqrt{uvw}}\,u\,dv\,dw.$$

Hence, after the substitution and multiplying by $\frac{4\sqrt{uvw}}{abc}$, (1) becomes

$$(b^2-c^2)\,u\,dv\,dw + (c^2-a^2)\,v\,dw\,du + (a^2-b^2)\,w\,du\,dv = 0 \ldots (3),$$

with the relation

$$u + v + w = 1 \ldots\ldots\ldots (4).$$

Differentiate (3); then, since

$$b^2 - c^2 + c^2 - a^2 + a^2 - b^2 = 0,$$

we get

$$(b^2-c^2)\,ud\,(dv\,dw) + (c^2-a^2)\,vd\,(dw\,du) + (a^2-b^2)\,wd\,(du\,dv) = 0 \ldots (5).$$

Now this is satisfied by the assumptions

$$dv\,dw = \frac{1}{f}, \quad dw\,du = \frac{1}{g}, \quad du\,dv = \frac{1}{h} \ldots\ldots\ldots (B),$$

f, g, h, being constants.

But from (4) we deduce

$$du + dv + dw = 0 \ldots\ldots\ldots (6),$$

and (B) gives

$$du = f\,du\,dv\,dw, \quad dv = g\,du\,dv\,dw, \quad dw = h\,du\,dv\,dw.$$

Hence $$f + g + h = 0 \ldots\ldots\ldots (7),$$

which establishes a relation among the otherwise arbitrary constants f, g, h.

Now (B) implies the existence of two linear equations in u, v, w. Hence, a *particular solution* of (1) is two linear equations connecting the three variables. But the given equation (4) is linear; hence the solution in question is the one congruent to the problem.

To find the other relation in u, v, w, eliminate the differentials from (3) by means of (B), and there is

$$(b^2 - c^2)\frac{u}{f} + (c^2 - a^2)\frac{v}{g} + (a^2 - b^2)\frac{w}{h} = 0 \ldots\ldots (8).$$

Equations (4) and (5), with the relation (7), contain the complete solution of the problem. It is obvious that the apparent want of homogeneity of (B) is wholly immaterial.

Keeping in mind the values of u, v, w, given by (A), we see that the geometrical interpretation of (8) is, every line of curvature on an ellipsoid lies on a conical surface of the second order, of which the vertex is the centre of the ellipsoid.

To determine the constants, let the line of curvature pass through a point, for which the values of u, v, w, are u_1, v_1, w_1, we have

$$(b^2 - c^2)\frac{u_1}{f} + (c^2 - a^2)\frac{v_1}{g} + (a^2 - b^2)\frac{w_1}{h} = 0,$$

$$f + g + h = 0.$$

Hence, after a slight reduction,

$$(b^2 - c^2)\,u_1\frac{g}{f} + (c^2 - a^2)\,v_1\frac{f}{g} - (a^2 - b^2)\,w_1 + (b^2 - c^2)\,u_1 + (c^2 - a^2)\,v_1 = 0 \ldots\ldots (9),$$

a quadratic in $\frac{g}{f}$, of which the roots are real and of unlike signs. This is obvious, for u_1, v_1, are essentially positive, and a, b, c, being in order of magnitude, the signs of $b^2 - c^2$ and $c^2 - a^2$ are opposite. Similarly, $\frac{g}{h}$ is determined by a quadratic, whose roots

are always real and of opposite signs. Thus two lines of curvature pass through every point on the surface of the ellipsoid.

Let us now consider the envelope of the surfaces represented by (8).

Differentiating (7) and (8) for f, g, h, we get

$$(b^2-c^2)\frac{u}{f^2}\,df+(c^2-a^2)\frac{v}{g^2}\,dg+(a^2-b^2)\frac{w}{h^2}\,dh=0\ldots\ldots(10),$$

$$df+dg+dh=0\ \ldots\ldots(11).$$

l being an indeterminate factor, we may put

$$l=(b^2-c^2)\frac{u}{f^2},\quad l=(c^2-a^2)\frac{v}{g^2},\quad l=(a^2-b^2)\frac{w}{h^2};$$

whence, taking the values of f, g, h, to substitute them in (8), we deduce

$$\sqrt{b^2-c^2}\sqrt{u}+\sqrt{c^2-a^2}\sqrt{v}+\sqrt{a^2-b^2}\sqrt{w}=0\ldots\ldots(12).$$

As the signs of the radicals are independent, this represents four planes; but c^2-a^2 is negative. Hence the possible part of these planes is their traces on the plane of xz, for which $v=0$. Thus we get the two straight lines

$$\sqrt{b^2-c^2}\sqrt{u}+\sqrt{a^2-b^2}\sqrt{w}=0\ \ldots\ldots(13),$$

and for the points where they meet the ellipsoid,

$$u+w=1\ \ldots\ldots\ \ldots\ldots\ldots\ (14),$$

whence

$$u_1=\frac{a^2-b^2}{a^2-c^2},\quad v_1=0,\quad w_1=\frac{b^2-c^2}{a^2-c^2}\ \ldots\ (15).$$

These values belong to the umbilici of the ellipsoid; a result easily anticipated. When they are introduced in (9), it becomes

$$\frac{g^2}{f^2}=0,\quad \text{and similarly}\quad \frac{g^2}{h^2}=0.$$

Hence (8) reduces to

$$v=0\ \ldots\ldots\ldots\ldots(16),$$

and represents the principal section of the ellipsoid, which passes through the greatest and least axes. In this case then, as our analysis would lead us to anticipate, the lines of curvature coincide; a result which, although well known, seems not very accurately demonstrated by Leroy. After having shown (p. 309 of the second edition) that the two *directions* of curvature coincide at the umbilical points, he proceeds to integrate, and passes from $\frac{dy}{dx}=0$ to $y=h$, and thence, determining the constant, to $y=0$; which

last represents the line of curvature sought. But $\frac{dy}{dx}$ has been shown to have the value 0, only for the umbilical points, and we are therefore not at liberty to pass by integration from these to any other points at which this may not hold. Were the process legitimate, it would lead to the strange conclusion, that the lines of curvature through an umbilicus are necessarily plane curves.

As there appears to be still some difficulty with regard to the theory of these singular points, we may enquire whether, in order to determine the lines of curvature through any point whatever, more is requisite than to substitute its co-ordinates in the general equation of the lines of curvature, and thus to get two values for the arbitrary constant; whether the result can ever be indeterminate, except when the lines, as at the extremity of an axis of revolution, are so in reality. In this view we see at once, that the process given by Leroy after Poisson for determining the *directions* of curvature at an umbilicus, is simply the ordinary method for ascertaining the position of the branches of any curve at a multiple point; and that the result arrived at, is not that more than two lines of curvature pass through an umbilicus, but that every point which, with reference to the surface, is umbilical, is, with reference to the lines of curvature, a multiple, or more generally a singular point. These suggestions may, perhaps, show how we must determine the lines of curvature which pass through an umbilicus, a problem distinct from that solved by Leroy of finding the directions of curvature.

Many curious properties may be deduced from the equations we have arrived at. Thus, if we take on two concentric and confocal ellipsoids, a series of pairs of corresponding points, (such as are spoken of in the enunciation of Ivory's theorem,) and if the locus of the points on one of the ellipsoids is a line of curvature, then that of those on the other is so too. Again, the traces on the tangent planes at the extremities of the three axes, made by one of the cones represented by (8), are an ellipse and two hyperbolas respectively. The areas of this ellipse, and of the ellipses conjugate to the two hyperbolas, are so related that their continual product is constant for the same ellipsoid, and for all ellipsoids of the same volume. The method of demonstrating these two theorems is so obvious, that it seems unnecessary to enter more fully on either.

It still remains to be shown how we pass from (8) to the projections of the lines of curvature on the co-ordinate planes. The symmetry of the problem is destroyed by the transition; but as it is in this shape that the results are commonly exhibited, we shall dwell rather more upon it than would otherwise have been necessary.

Putting $C = a^2 - b^2$, $B = c^2 - a^2$, $A = b^2 - c^2$, and eliminating u, v, w, successively between (4) and (8), there result

$$\left.\begin{aligned} \left(\frac{C}{h} - \frac{A}{f}\right) u - \left(\frac{B}{g} - \frac{C}{h}\right) v &= \frac{C}{h} \\ \left(\frac{A}{f} - \frac{B}{g}\right) v - \left(\frac{C}{h} - \frac{A}{f}\right) w &= \frac{A}{f} \\ \left(\frac{B}{g} - \frac{C}{h}\right) w - \left(\frac{A}{f} - \frac{B}{g}\right) u &= \frac{B}{g} \end{aligned}\right\} \ldots\ldots (17).$$

Put

$$\frac{C}{h} - \frac{A}{f} = k, \quad \frac{B}{g} - \frac{C}{h} = l, \quad \frac{A}{f} - \frac{B}{g} = m \ldots\ldots (18).$$

Then $kf - lg = C\left(\frac{f+g}{h}\right) - A - B = -(A + B + C) = 0.$

Thus we get the relations

$$\left.\begin{aligned} kf - lg &= 0 \\ lh - mf &= 0 \\ mg - kh &= 0 \end{aligned}\right\} \ldots\ldots\ldots (19).$$

Hence, $\frac{A}{mf} - \frac{B}{mg} = 1 = \frac{1}{h}\left(\frac{A}{l} - \frac{B}{k}\right)$,

and consequently

$$\left.\begin{aligned} h &= \frac{A}{l} - \frac{B}{k} \\ \text{similarly, } g &= \frac{C}{m} - \frac{A}{l} \\ f &= \frac{B}{k} - \frac{C}{m} \end{aligned}\right\} \ldots\ldots\ldots (20).$$

By means of (18) and (20) the equations (17) become

$$\left.\begin{aligned} ku - lv &= \frac{Clk}{Ak - Bl} \\ mv - kw &= \frac{Amk}{Bm - Ck} \\ lw - mu &= \frac{Blm}{Cl - Am} \end{aligned}\right\} \ldots\ldots (21);$$

which, restoring their values to u, v, w, may be written

$$\frac{y^2}{b^2} = \frac{k}{l}\left\{\frac{x^2}{a^2} - \frac{C}{A\frac{k}{l} - B}\right\}, \text{ \&c. } \ldots\ldots\ldots (22).$$

Put $\frac{k}{l}\frac{b^2}{a^2} = m$; $\therefore \frac{k}{l} = m\frac{a^2}{b^2}$, and then we get

$$y^2 = m\left\{x^2 - \frac{a^2}{b^2}\frac{a^2 - b^2}{m\frac{a^2}{b^2}(b^2 - c^2) - (c^2 - a^2)}\right\},$$

or $$y^2 = m\left\{x^2 - \frac{a^2(a^2 - b^2)}{ma^2(b^2 - c^2) - b^2(c^2 - a^2)}\right\} \ldots(23),$$

with similar equations for the projections on the other co-ordinate planes. This result is identical with the known one in Leroy, p. 304, or Hymers, p. 201.

It is hoped that the novelty of treating symmetrically a non-integrable equation in three variables, will be admitted as an excuse for the length to which this paper has extended itself.

X.—ON THE THEORY OF MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES.

ALTHOUGH it is usual to illustrate the Theory of Maxima and Minima of Functions of one Variable by a reference to the properties of curve lines, I do not remember that any similar illustration has been used in treating of Maxima and Minima of Functions of two Variables. As far as the first condition is concerned, the illustrations will be the same in both cases, but the second or Lagrange's condition in functions of two variables has nothing analogous in the case of two variables. The geometrical explanation gives a very distinct idea of its meaning, and on that account I think that a notice of it may be useful to the student.

Let us briefly consider how the condition arises analytically, and afterwards proceed to the geometrical interpretation of the various steps.

If $z = f(x, y)$ be a maximum or minimum, and z_1 be the value of z when $x + h$, $y + k$ are substituted for x and y, z_2 the value when $x - h$, and $y - k$ are substituted; then for a maximum we must have

$$z_1 < z \text{ and } z_2 < z,$$

$$\text{or } z_1 - z < 0, \quad z_2 - z < 0;$$

and for a minimum,

$$z_1 - z > 0, \quad z_2 - z > 0;$$

Now

$$z_1 - z = \frac{dz}{dx}h + \frac{dz}{dy}k + \frac{1}{1.2}\left(\frac{d^2z}{dx^2}h^2 + 2\frac{d^2z}{dx\,dy}hk + \frac{d^2z}{dy^2}k^2\right) + \&c.$$

and

$$z_2 - z = -\left(\frac{dz}{dx}h + \frac{dz}{dy}k\right) + \frac{1}{1.2}\left(\frac{d^2z}{dx^2}h^2 + 2\frac{d^2z}{dx\,dy}hk + \frac{d^2z}{dy^2}k^2\right) - \&c.$$

Now either for a maximum or minimum it appears that $z_1 - z$ and $z_2 - z$ must be of the same sign, and as h and k can be assumed so small that the sign of the whole series after them depends on the sign of the terms involving their first powers, and as these are necessarily of opposite signs. In the two series $z_1 - z$ and $z_2 - z$ cannot be of the same sign, unless the terms involving the first powers of h and k vanish, or

$$\frac{dz}{dx}h + \frac{dz}{dy}k = 0,$$

which, as h and k are independent, involves the two conditions

$$\frac{dz}{dx} = 0, \qquad \frac{dz}{dy} = 0 \ldots\ldots\ldots (A).$$

But $z_1 - z$ and $z_2 - z$ must both of them remain of the same sign, whatever value we assign to h and k, and therefore the first remaining term of the series must remain constantly of the same sign, whatever values we assign to h and k. That is to say, the expression

$$\frac{d^2z}{dx^2}h^2 + 2\frac{d^2z}{dx\,dy}hk + \frac{d^2z}{dy^2}k^2,$$

must not pass from $+$ to $-$, or conversely, from changes in the magnitude or sign of h and k. Let $k = mh$, then the expression becomes

$$h^2\left(\frac{d^2z}{dx^2} + 2\frac{d^2z}{dx\,dy}m + \frac{d^2z}{dy^2}m^2\right);$$

and, as h^2 is essentially positive, the sign of

$$\frac{d^2z}{dy^2}m^2 + 2\frac{d^2z}{dx\,dy}m + \frac{d^2z}{dx^2},$$

must not change. Now this expression can only change sign by passing through 0; and, in order that this may never happen from any change in m, the value of m derived from that expression equated to 0 must be impossible.

The solution of the equation gives, if we put

$$\frac{d^2z}{dx^2} = r, \quad \frac{d^2z}{dx\,dy} = s, \quad \frac{d^2z}{dy^2} = t,$$

$$m = -\frac{s}{t} \pm \frac{\sqrt{s^2 - rt}}{t};$$

and, in order that this may be impossible, we must have

$$s^2 < rt,$$

or $$\left(\frac{d^2z}{dx\,dy}\right)^2 - \left(\frac{d^2z}{dx^2}\right)\left(\frac{d^2z}{dy^2}\right) = 0 \ldots\ldots \text{(B)}.$$

This is Lagrange's condition.

Let us now consider the geometrical interpretation of the various steps.

$$z = f(x, y)$$

represents the equation to a surface, and the conditions (A) imply that the tangent plane must be parallel to the plane of xy, since the equation to the tangent plane becomes, under those conditions,

$$z - z_1 = 0.$$

The assumption of $k = mh$ establishing a relation between the increments of x and y, independent of z, corresponds to taking a section of the surface by a plane perpendicular to xy, its trace being inclined to the axis of x at an angle whose tangent is m. For the interpretation of the condition B, we must have recourse to the expression for the radius of curvature of a normal section.

If ρ be the radius of curvature, we have, under the condition that $\frac{dz}{dx} = 0, \quad \frac{dz}{dy} = 0,$

$$\rho = \frac{1 + m^2}{r + 2sm + tm^2}.$$

Now the sign of this depends only on that of the denominator, since the numerator equated to zero gives only impossible values for m, but the denominator is exactly the quantity, the sign of which was considered before. Therefore the condition (B), considered geometrically, implies that the sign of the radius of curvature of a normal section shall not change, that is, that every section at the point under consideration must be either convex or concave, but must never pass from one species of curvature to the other. The reason of this is obvious, as the ordinate which is a maximum for a concave section is a minimum for a convex one, and *vice versâ*. It might perhaps be advantageous to have a name appropriated to those points of surfaces for which the conditions (A) hold, but not the condition (B). Such points, though they do not possess the property of being absolute maxima or minima, are yet for many purposes quite as much worthy of attention, since we have frequently to consider geometrically only the fact that the ordinate is stationary for a short space, or that the tangent plane is then perpendicular to the ordinate. Thus, for instance, in investigating the properties of principal diameters in surfaces of the second order, we have only to consider the first condition, and do not require to pay attention to the second. Perhaps the name of "Stationary Points" would be sufficiently distinctive.

D.

XI.—NOTE ON THE CALCULATION OF FORMULÆ IN DIFFRACTION.

THE following transformation is occasionally found useful in calculating the formulæ which occur in the investigation of phenomena of diffraction. These generally depend on finding the value of the definite integral

$$\int_b^a dx\, f(x) \sin \frac{2\pi}{\lambda} \{vt - \psi(x)\},$$

in the form of

$$A \sin \frac{2\pi}{\lambda} vt + B \cos \frac{2\pi}{\lambda} vt;$$

when the intensity of illumination is proportional to $A^2 + B^2$. Now

$$A = \int_b^a dx\, f(x) \cos\left(\frac{2\pi}{\lambda}\psi(x)\right), \quad B = -\int_b^a dx\, f(x) \sin\left(\frac{2\pi}{\lambda}\psi(x)\right);$$

putting for the cosine and sine their exponential values, squaring and adding, we find

$$A^2 + B^2 = \left(\int_b^a dx\, f(x)\, \varepsilon^{\frac{2\pi}{\lambda}\psi(x)\sqrt{-1}}\right)\left(\int_b^a dx\, f(x)\, \varepsilon^{-\frac{2\pi}{\lambda}\psi(x)\sqrt{-1}}\right).$$

In practice, $\psi(x)$ is generally and may always be made x; and if one of the limits is ∞, it is to be observed that we must consider $\varepsilon^{x\sqrt{-1}}$ to be equal to 0 when $x = \infty$, that being the average of all its values. In such cases the formula is generally considerably simplified, especially when $f(x)$ is constant.

As an example of the application of this formula, let us take the problem of finding the intensity at the centre of the shadow of a small circular disk.

Let A (fig. 7.) be the centre of the disk, AC its radius $= c$, B the centre of the shadow, $AB = a$, $BC = b$, $PB = r$, $AP = u$.

The vibration at B due to the action of an annulus whose radius is u and distance from B r, is

$$C \frac{2\pi u\, du}{r} \sin \frac{2\pi}{\lambda} (vt - r);$$

and the whole vibration excited at B is given by

$$2\pi C \int_c^\infty \frac{u\, du}{r} \sin \frac{2\pi}{\lambda} (vt - r).$$

But as $r^2 = a^2 + u^2$, and therefore $u\, du = r\, dr$, this may be changed into

$$2\pi C \int_b^\infty dr \sin \frac{2\pi}{\lambda} (vt - r),$$

b being the value of r corresponding to $u = c$.

By our previous transformation we find for the intensity of the illumination the expression

$$4\pi^2 C^2 \left(\int_b^\infty dr\, \varepsilon^{\frac{2\pi}{\lambda} r\sqrt{-1}} \right) \left(\int_b^\infty dr\, \varepsilon^{-\frac{2\pi}{\lambda} r\sqrt{-1}} \right),$$

which, on effecting the operations indicated, becomes

$$4\pi^2 C^2 \frac{\lambda}{2\pi\sqrt{-1}} \varepsilon^{\frac{2\pi}{\lambda} r\sqrt{-1}} \cdot \frac{-\lambda}{2\pi\sqrt{-1}} \varepsilon^{-\frac{2\pi}{\lambda}\sqrt{-1}}$$

$$= 4\pi^2 C^2 \frac{\lambda^2}{4\pi^2} = C^2\lambda^2,$$

which, being independent of the radius of the disk, shews that the intensity is the same as if the disk were removed.

H. T.

XII.—MATHEMATICAL NOTES.

1. The area of a polygon of a given number of sides, circumscribing a given oval figure, will be the least possible when each side is bisected in the point of contact.

This elegant proposition, given in the *Senate-House Problems* for 1836, may be easily demonstrated as follows:—

Let AB, BC, CD, be consecutive sides of the polygon. Produce AB, DC, to meet in E; then BC must, by the condition of the minimum, be in such a position that EBC is a maximum.

Refer the oval to EA, ED, for axes, then the equation to the tangent BC is

$$y'\,dx - x'\,dy = y\,dx - x\,dy,$$

y and x being the co-ordinates of the point of contact P.

Put $x' = 0$;

$$\therefore\ y_0\,dx = y\,dx - x\,dy,$$

$$\text{and so}\quad -x_0\,dy = y\,dx - x\,dy.$$

$$\text{Also area of EBC} = \tfrac{1}{2}x_0 y_0 \sin \text{E}.$$

Hence, $\dfrac{(y\,dx - x\,dy)^2}{dx\,dy}$ is a maximum, (the minus sign is immaterial).

Differentiate, considering x as independent; then

$$\frac{y\,dx - x\,dy}{dx\,dy}\, d^2y \left(\frac{y\,dx - x\,dy}{dy} + 2x \right) = 0.$$

The last factor only gives a solution;

$$\therefore\ x = -\tfrac{1}{2}\frac{y\,dx - x\,dy}{dy} = \tfrac{1}{2}x_0,$$

that is, PM being parallel to EC, $EM = \frac{1}{2}EB$, and $\therefore$ BP = PC, or BC is bisected in the point of contact P. The same is true of any other side, and therefore every side is bisected in the point of contact.

ε.

2. The following method of investigating the conditions that a straight line and a plane may be at right angles to one another, is easier than the geometrical one which is ordinarily given in books on Analytical Geometry, (see *Leroy's* Géométrie, p. 28).

Let the equation to any plane be

$$Ax + By + Cz = D \ldots\ldots (1),$$

and let the equations to a straight line at right angles to it be

$$\left.\begin{aligned} x &= mz + p \\ y &= nz + q \end{aligned}\right\} \ldots\ldots\ldots (2).$$

Let the equations to any straight line lying within the plane (1), and passing through the intersection of (1) and (2), be

$$\left.\begin{aligned} x &= m'z + p' \\ y &= n'z + q' \end{aligned}\right\} \ldots\ldots\ldots (3).$$

Then, since (2) and (3) must be at right angles to each other, we have

$$1 + mm' + nn' = 0 \ldots\ldots (4).$$

But, since (3) coincides with (1), we have

$$Am' + Bn' + C = 0;$$

and therefore, multiplying the equation (4) by (B), and introducing this last relation, we have

$$B + Bmm' - n(Am' + C) = 0,$$

$$\text{or}\quad B - nC + m'(Bm - An) = 0;$$

and this being true for all values of the indeterminate quantity m', we have, as the required conditions,

$$B - nC = 0, \quad \text{or } n = \frac{B}{C},$$

$$\text{and}\quad Bm - An = 0, \quad \text{or } m = \frac{An}{B} = \frac{A}{C}.$$

w.

3. Let p, p', be two forces into which a given system on a rigid body may be resolved, a, θ, their least distance, and inclination of their directions; $pp'\,a \sin\theta$ is invariable. (*Senate-House*, 1833.)

Let the line a meet the directions of p and p' in P and P′ respectively. At P apply two forces equal and parallel to p', and

opposite each other. Thus the system of forces is replaced by the couple $p'a$, and by the force at P, which is the resultant of p and p'. Resolve this, the resultant, along the axis of the couple and in its plane. Then the former component can arise only from the resolved part of p, as p' is wholly in the plane of the couple. Also, as the shortest distance is perpendicular to both lines, it follows that the arm of the couple is perpendicular at P to the plane which contains the two forces p and p'. Hence θ, their mutual inclination, is that of p on the plane of the couple, and therefore $p \sin \theta$ is the part of the general resultant resolved along the axis of the couple. Then, if the general resultant makes an angle ϕ with the axis, we have in the usual notation

$$pp'a \sin \theta = \mathrm{GR} \cos \phi = \mathrm{R.G} \cos \phi.$$

Now $\mathrm{G} \cos \phi$, as is known, or as may be easily shown, $= \mathrm{G}_1$, the minimum maximorum moment of the system;

$$\text{therefore } pp'a \sin \theta = \mathrm{G}_1\mathrm{R},$$

which is constant.

ε.

CORRIGENDA.

In Vol. 2, p. 27, line 4, *read* 1 *instead of* (*a*).

In p. 56, line 14, *should be written*

$$\frac{dy_{\tau+2}}{dy_{\tau+1}} = \infty, \quad \frac{dy_{\tau+3}}{dy_{\tau+1}} = \infty, \quad \ldots \frac{dy_n}{dy^{\tau}_{+1}} = \infty.$$

In p. 58, line 14 from the bottom, change $\frac{dy_{\tau+2}}{dy_\tau}$ into $\frac{dy_{\tau+2}}{dy_{\tau+1}}$; and *read, instead of* line 7 from the bottom,

$$\frac{dy_2}{dy_1} = \infty, \quad \frac{dy_3}{dy_2} = \infty, \quad \frac{dy_4}{dy_3} = \infty, \quad \ldots \frac{dy_n}{dy_{n-1}} = \infty.$$

In p. 95, line 2, *read* $\mathrm{G}\, a^{x-1} \Sigma \frac{u_x}{a_x}$ *for* $\mathrm{G}.a^{x-1} \varepsilon \frac{u_x}{a_x}$.